LARGE VISCOSITY SOLUTIONS FOR SOME FULLY NONLINEAR EQUATIONS

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Abstract. We study existence, uniqueness and asymptotic behavior near the boundary of solutions of the problem

$$\begin{cases} -F(D^2u) + \beta(u) = f & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N > 1$, F is a fully nonlinear elliptic operator and β is an increasing continuous function. Assuming that β satisfies the Keller-Osserman condition, we obtain existence results which apply to $f \in L^\infty_{loc}(\Omega)$ or f having only local integrability properties where viscosity solutions are well defined, i.e. $f \in L^N_{loc}(\Omega)$. Besides, we find the asymptotic behavior near the boundary of solutions of (P) for a wide class of functions $f \in \mathcal{C}(\Omega)$. Based in this behavior, we also prove uniqueness.

1. Introduction

In this paper we study the following problem

$$-F(D^2u) + \beta(u) = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = +\infty \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N > 1$, F is a fully nonlinear elliptic operator and β is an increasing continuous function. In general, solutions of (1.1) that verify (1.2) are known as *large solutions* due to the explosive boundary condition $u = +\infty$, that is interpreted as

$$u(x) \to +\infty \text{ as } \delta(x) \to 0,$$

where we have introduced the following notation that we use from now on

$$\delta(x) := \text{dist}(x, \partial\Omega).$$

Since a Comparison Principle holds, the inequality

$$u \geq v \text{ in } \Omega,$$

is satisfied for any other solution v of (1.1) with bounded boundary values. Thus, sometimes large solutions also are called maximal solutions.

Our main goal here is to obtain different solvability situations for (1.1)-(1.2), find the asymptotic behavior near the boundary for viscosity solutions and establish uniqueness results. In this way, we seek to extend some results in [13]. In particular, there was proved an existence result for the special choice $\beta(t) = |t|^{p-1}t$, $p > 1$.

We recall that when F is the Laplacian and $f \equiv 0$, a first study known about problem (1.1)-(1.2) is due to Bieberbach [4], whereas existence of solutions with β monotone, was established

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by Keller [18] and Osserman [23] who found a necessary and sufficient condition on the growth at infinity of β in order to guarantee that such solutions exist. This is the well-known Keller-Osserman condition

(β_0) $$\int^{\infty} \frac{ds}{\sqrt{\mathbf{B}(s)}} < +\infty,$$

where $\mathbf{B}(t) := \int_0^t \beta(s)\,ds$. In order to see more related results, we refer the reader to the survey [24] and references therein.

On the operator F we assume uniform ellipticity, that is:

$$\mathcal{M}^-_{\lambda,\Lambda}(M-N) \le F(M) - F(N) \le \mathcal{M}^+_{\lambda,\Lambda}(M-N) \quad \text{for all} \quad N, M \in \mathcal{S}_N,$$

with $\mathcal{S}_N$ denoting the space of all real symmetric $N\times N$ matrices, and $F(0) = 0$. Here, $0 < \lambda \le \Lambda$ and, $\mathcal{M}^-_{\lambda,\Lambda}$ and $\mathcal{M}^+_{\lambda,\Lambda}$ are the Pucci's extremal operators defined as in [6] by

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i \quad \text{and} \quad \mathcal{M}^+_{\lambda,\Lambda}(M) = \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i,$$

where $e_i = e_i(M)$ are the eigenvalues of M. These operators are extremal in the sense that

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) = \inf_{A\in\mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AD^2u) \quad \text{and} \quad \mathcal{M}^+_{\lambda,\Lambda}(D^2u) = \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AD^2u),$$

where $\mathcal{A}_{\lambda,\Lambda} = \{A \in \mathcal{S}_N : \lambda|\xi|^2 \le a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2, \ \text{ for all } \ \xi \in \mathbb{R}^N\}$.

Fully nonlinear elliptic operators appear, for example, in problems of optimal control for stochastic differential equations, see [14]. On the other hand, when F is the Laplacian, the problem (1.1)-(1.2) is related with super-diffusions, see for example [12] and [20]. Hence, our problem is related with optimal control for some stochastic differential equation involving super-Brownian motion. This is an interesting thing to explore from the probability point of view and, as far as we know, the problem still remains open.

Before continuing, we give our notion of solutions on which we are interested.

Definition 1.1. *Assume that $f \in L^p_{loc}(\Omega)$. We call $u \in \mathcal{C}(\Omega)$ an L^p-viscosity subsolution (supersolution) of (1.1) if for all $\varphi \in W^{2,p}_{loc}(\Omega)$ and a point $x_0 \in \Omega$ at which $u - \varphi$ has a local maximum (minimum) one has*

$$\operatorname{ess}\liminf_{x\to x_0}(-F(D^2\varphi(x)) + \beta(u(x)) - f(x)) \le 0$$

$$(\operatorname{ess}\limsup_{x\to x_0}(-F(D^2\varphi(x)) + \beta(u(x)) - f(x)) \ge 0).$$

Moreover, u is an L^p-viscosity solution of (1.1) if it is both an L^p-viscosity subsolution and an L^p-viscosity supersolution. In particular, if additionally u satisfies (1.2), then we say that u is an L^p-viscosity large solution of (1.1).

Our first theorem is about existence of an L^∞-viscosity solution of problem (1.1)-(1.2), and it is given for the case $f \in L^\infty_{loc}(\Omega)$. In order to put in perspective our result, we consider the following conditions

(β_1) β is a nondecreasing continuous function.

(β_2) β is odd.

Theorem 1.1. *If $f \in L^\infty_{loc}(\Omega)$, $f \ge 0$ a.e. and β satisfies (β_0) and (β_1), then (1.1) possesses at least one L^∞-viscosity large solution such that $u \ge 0$ in Ω. If $f \in L^\infty_{loc}(\Omega)$ is such that $f \ge g$ for some $g \in L^\infty(\Omega)$ and β satisfies (β_0), (β_1) and (β_2), then (1.1) possesses at least one L^∞-viscosity large solution.*

We recall that if $f \in \mathcal{C}(\Omega)$, then, by regularity theory, the solution u of (1.1)-(1.2) found in the previous theorem indeed is a $\mathcal{C}$-viscosity solution of (1.1) that verifies (1.2), which here we call $\mathcal{C}$-viscosity large solution.

Next theorem deals with the case $f \in L^N_{loc}(\Omega)$, where we impose an extra assumption on β:

(β_3) $\beta \in \mathcal{C}^1(0,\infty)$ is such that $\dfrac{\beta(t)}{t^q}$ is increasing for some $q > 1$.

Theorem 1.2. *If $f \in L^N_{loc}(\Omega)$, $f \geq 0$ a.e. and β satisfies (β_0), (β_1) and (β_3), then (1.1) possesses at least one L^N-viscosity large solution such that $u \geq 0$ in Ω. If $f \in L^N_{loc}(\Omega)$ is such that $f \geq g$ for some $g \in L^N(\Omega)$, and β satisfies (β_0), (β_1), (β_2) and (β_3), then the equation (1.1) possesses at least one L^N-viscosity large solution.*

Notice that this theorem is new even when F is the Laplacian, moreover the upper bound used in the proof can not be obtained by the standard construction of explosive barrier function. For the proof we use an Alexandroff-Bakelman-Pucci estimate and a cut-off function. Continuing with the known results, when F is the Laplacian, the p-Laplacian or some other more general second elliptic operator with divergence form, the asymptotic behavior of first order near the boundary and uniqueness of solutions of (1.1)-(1.2), with $f \equiv 0$, already has been extensively studied in the literature for a wide class of functions β, see for example [1, 2, 22, 15]. In [17] large solutions were studied for equations involving the infinity Laplacian operator and $f \equiv 0$. When $f \not\equiv 0$ the above theorem is obtained also for singular or degenerate fully nonlinear operators in the case $\beta(t) = |t|^{p-1}t$, see [10]. Therefore, a natural question is if the above theorem can be extended to more general fully nonlinear operator, possibly nondegenerate. A pioneer work in this direction is [11], see also [22].

Returning to the blow-up rate, in [2] the asymptotic behavior was found assuming that β is a nonnegative function on $[0,\infty[$ that verifies (β_0), (β_1) and

(β_4) $\liminf_{t\to\infty} \dfrac{\psi(\rho t)}{\psi(t)} > 1$, for all $\rho \in]0,1[$,

where

$$\psi(t) := \int_t^\infty \frac{ds}{\sqrt{2\mathbf{B}(s)}}, \quad \text{for all } t \text{ sufficiently large}, \tag{1.3}$$

and β is a locally Lipschitz-continuous function for all $t \geq 0$, which is nondecreasing for all t sufficiently large (see also [3], where (β_4) is assumed, with $\beta \in \mathcal{C}^1(0,\infty)$, $\beta(t) > 0$ and $\beta'(t) > 0$ for all t sufficiently large). Note that (β_3) implies (β_4), so (β_4) is a more weak assumption that (β_3). On the other hand, uniqueness can be obtained by using the asymptotic behavior near the boundary and the following additional assumption on β

(β_5) $\dfrac{\beta(t)}{t}$ is increasing for all $t > 0$.

However, far as we know, the asymptotic behavior near the boundary and uniqueness of large solutions for fully nonlinear operators have not yet been studied. In this way, next result represents a first effort in order to find results about the asymptotic behavior near the boundary and uniqueness of solutions for (1.1)-(1.2). In particular, for $\eta \in [0,1[$ and $f \in \mathcal{C}(\Omega)$ such that

$$\lim_{\delta(x)\to 0} \frac{f(x)}{\beta\big(\phi\big(\sqrt{(1-\eta)F(A)^{-1}}\,\delta(x)\big)\big)} = \eta, \tag{1.4}$$

where $A := \mathrm{diag}[0,0,\ldots,1]$ and

$$\phi(\delta) := \psi^{-1}(\delta), \quad \text{for all } \delta > 0 \text{ sufficiently small},$$

we are able of finding the blow-up rate for large viscosity solutions of (1.1) for a wide class of nonlinearities β such as shows the next theorem.

Theorem 1.3. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N > 1$, with $\partial\Omega$ of class $\mathcal{C}^2$, let $\eta \in [0,1[$ and let $f \in \mathcal{C}(\Omega)$ a nonnegative function that verifies (1.4). If β satisfies (β_0), (β_1) and (β_4) then every nonnegative $\mathcal{C}$-viscosity large solution u of (1.1) verifies*

$$\lim_{\delta(x)\to 0} \frac{u(x)}{\phi\big(\sqrt{(1-\eta)F(A)^{-1}}\,\delta(x)\big)} = 1.$$

Moreover, if in addition (β_5) holds, then (1.1) admits a unique nonnegative $\mathcal{C}$-viscosity large solution.

In the proof we use some ideas introduced in [21, 11], where was considered the particular choice $\beta(t) = t^p$, $p > 1$. It is based in the construction of suitable sub- and supersolutions.

In order to illustrate our theorem above, we show the following examples in case $\eta = 0$:

(1) If $\beta(t) = t^p$, $p > 1$, and $\lim_{\delta(x)\to 0} f(x)(\delta(x))^{-\alpha} \leq C$, for some constant $C \geq 0$, where $0 < \alpha < 2p(p-1)^{-1}$, then every $\mathcal{C}$-viscosity large solution of (1.1) satisfies

$$\lim_{\delta(x)\to 0} \left(u(x) \left(\frac{2(p+1)}{F(A)(p-1)^2} \right)^{\frac{1}{p-1}} (\delta(x))^{\frac{2}{p-1}} \right) = 1. \tag{1.5}$$

(2) If $\beta(t) = e^t$, and $\lim_{\delta(x)\to 0} f(x)(\delta(x))^{-\alpha} \leq C$, for some constant $C \geq 0$, where $0 < \alpha < 2$, then every $\mathcal{C}$-viscosity large solution of (1.1) satisfies

$$\lim_{\delta(x)\to 0} \left(u(x) \left(\log \frac{2F(A)}{(1-\eta)(\delta(x))^2} \right)^{-1} \right) = 1. \tag{1.6}$$

Note that these examples are consistent with the known results to problem (1.1)-(1.2) with F being the Laplacian, in whose case $F(A) \equiv 1$. Moreover, if in example (1) (or example (2)) we consider $F = \mathcal{M}^+_{\lambda,\Lambda}$ or $F = \mathcal{M}^-_{\lambda,\Lambda}$, then rate obtained here coincides with that in (1.5) (or (1.6)), when one replaces $F(A)$ respectively by Λ or λ. Also, in the proof one can observe that if $f \in \mathcal{C}(\overline{\Omega})$, then it is possible to remove the restriction $f \geq 0$ for finding the blow-up rate, but not for uniqueness. See Section 3 for more details related with our examples. There also we show an extension of our result related on blow-up rate and uniqueness including a case where f not verify (1.4).

2. Existence of large solutions

From now on we assume that Ω is a bounded open domain in $\mathbb{R}^N$, $N \geq 2$, with boundary of class $\mathcal{C}^2$. Also, for simplicity notational, from now on we will put $\mathcal{M}^- := \mathcal{M}^-_{\lambda,\Lambda}$ and $\mathcal{M}^+ := \mathcal{M}^+_{\lambda,\Lambda}$.

The first step in this section consists of solving the problem

$$\begin{cases} -F(D^2u) + \beta(u) = f & \text{in } \Omega, \\ u = n & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $f \in \mathcal{C}(\Omega)$ and $n \in \mathbb{N}$. Here u denotes a continuous viscosity solution of the problem (2.1).

We start with a key result in our reasoning, which is a comparison result.

Lemma 2.1 (**Comparison Lemma**). *Assume that β satisfies (β_1) on $\mathbb{R}^+$ and that $f \in \mathcal{C}(\Omega)$. If $u, v \in \mathcal{C}(\Omega)$ are respectively a $\mathcal{C}$-viscosity subsolution and a $\mathcal{C}$-viscosity supersolution of (1.1), and*

$$\limsup_{\delta(x)\to 0} \frac{u(x)}{v(x)} < 1, \tag{2.2}$$

then

$$u \leq v \quad \text{in } \Omega. \tag{2.3}$$

Proof. We argue by contradiction. If (2.3) is not true, from (2.2) and by continuity of u and v in Ω, we can suppose that there exists an open set $\Theta \subset\subset \Omega$ such that

$$u > v \quad \text{in } \Theta \quad \text{and} \quad u = v \quad \text{on } \partial\Theta.$$

with $(u - v) \in \mathcal{C}(\overline{\Theta})$. Hence

$$\beta(u) \geq \beta(v) \quad \text{in } \Theta.$$

On the other hand, since u is a $\mathcal{C}$-viscosity subsolution of (1.1) and v is a $\mathcal{C}$-viscosity supersolution of (1.1), we have that

$$F(D^2u) - F(D^2v) \geq 0 \quad \text{in } \Theta$$

in the $\mathcal{C}$-viscosity sense. Since F is uniformly elliptic, by applying Proposition 2.1 in [9], we get

$$\mathcal{M}^+(D^2(u-v)) \geq 0 \quad \text{in } \Theta$$

in the $\mathcal{C}$-viscosity sense and $u - v = 0$ on $\partial\Theta$. Then, bearing in mind that $(u-v) \in \mathcal{C}(\overline{\Theta})$, by the Alexandroff-Bakelman-Pucci maximum principle we obtain

$$u \leq v \quad \text{in } \Theta,$$

which is a contradiction. Therefore (2.3) holds. □

Remark 2.1. *Notice that the above Comparison Lemma also holds when we assume that* $u = v = n$ *on* $\partial\Omega$.

Lemma 2.2. *Assume that* $f \in C(\bar{\Omega})$ *and* β *satisfies* (β_0), (β_1) *and* (β_2). *Then for every* $n \in \mathbb{N}$ *there is a* C*-viscosity solution* $u \in \mathcal{C}^{1,\alpha}(\Omega)$ *of the problem (2.1).*

Proof. Note that under our assumptions one has

$$\lim_{t\to+\infty} \frac{\beta(t)}{t} = +\infty. \tag{2.4}$$

Indeed, in other case $\beta(t)/t$ would be bounded for all t sufficiently large. Hence, there would exists $t_0 > 0$ sufficiently large, such that

$$\beta(t) \leq C \quad \text{for all } 0 < t < t_0 \quad \text{and} \quad \beta(t) \leq Ct \quad \text{for all } t > t_0,$$

for some $C > 0$ independients on all $t > t_0$. Hence,

$$\mathbf{B}(t) = \int_0^t \beta(s)ds \leq C(1+t^2) \quad \text{for all } t > t_0$$

and

$$\int_{t_0}^t \frac{ds}{\sqrt{1+s^2}} \leq \sqrt{C}\int_{t_0}^t \frac{ds}{\sqrt{\mathbf{B}(s)}} \quad \text{for all } t > t_0.$$

But this leads to a contradiction with the Keller-Osserman condition (β_0), because the integral on the left side of the inequality above diverges to infinity as t tends to infinity.

In case $f \equiv 0$, we note that $\underline{v} \equiv 0$ is a subsolution of (2.1) whereas that $\overline{v} \equiv M$ is a supersolution for M large by (2.4).

In case $f \not\equiv 0$, since β is odd and satisfies (2.4), there exists a constant $M \geq \max\{n, \|f\|_{L^\infty(\Omega)}\}$ such that

$$-\beta(M) \leq f(x) \leq \beta(M), \quad \text{a. e.} \quad x \in \Omega.$$

In this way, $\underline{v} \equiv -M$ and $\overline{v} \equiv M$ are respectively a subsolution and a supersolution of the equation in (2.1). Now, let be γ a positive constant such that the function $r(t) = \gamma t - \beta(t)$ becomes increasing in the interval $[-M, M]$. Putting $v_0 = \underline{v}$ and by applying iteratively Theorem 1.1 of [8] we get for every $k \in \mathbb{N}$ a function $v_k \in \mathcal{C}(\overline{\Omega})$, which $\mathcal{C}$-viscosity problem

$$\begin{cases} -F(D^2v_k) + \gamma v_k = f + \gamma v_{k-1} - \beta(v_{k-1}) & \text{in } \Omega, \\ v_k = n & \text{on } \partial\Omega. \end{cases}$$

It follows from Remark 2.1 and the monotonicity of r, that the family $\{v_k\} \subset \mathcal{C}(\overline{\Omega})$ verifies

$$-M = \underline{v} \leq v_k \leq v_{k+1} \leq \overline{v} = M, \quad \forall k \in \mathbb{N}.$$

Since $\mathcal{C}$-viscosity solution are L^N- viscosity solution from Proposition 4.2 in [8], it follows that the set $\{v_k\}$ is a precompact subset of $\mathcal{C}(\overline{\Omega})$, which implies that a subsequence $\{v_k\}$ converges uniformly to u in $\bar{\Omega}$ by Proposition 2.9 of [6] conclude that u is a $\mathcal{C}$-viscosity solution of the problem (2.1). Finally, from the regularity results in [5] we find that $u \in \mathcal{C}^{1,\alpha}(\Omega)$. □

The following results will be useful to establish a priori estimates of viscosity solutions of the equation (1.1). For this purpose, we start with a version of Kato type inequality, which need only if the function f is negative in some point of the domain.

Lemma 2.3. *Assume that $u, f \in \mathcal{C}(\Omega)$ and β satisfies (β_2). If u is a viscosity solution of (1.1), then $|u|$ satisfies*

$$-\mathcal{M}^+(D^2|u|) + \beta(|u|) \le |f| \quad \text{in } \Omega, \tag{2.5}$$

in the $\mathcal{C}$-viscosity sense.

Proof. We first notice that $u^+ = \max\{u, 0\}$ is a subsolution of (1.1) with f^+ as right hand side, by standard viscosity solution argument, see Proposition 2.8 in [6]. In this way, since $F \le \mathcal{M}^+$, we obtain

$$-\mathcal{M}^+(D^2u^+) + \beta(u^+) \le f^+.$$

Also observe that since β is odd, then

$$-F(D^2(-u)) + \beta(-u) = f(-x).$$

Hence, since $F \le \mathcal{M}^+$ and $\mathcal{M}^-(D^2u) = -\mathcal{M}^+(-D^2u)$ and $\mathcal{M}^- \le \mathcal{M}^+$, we get

$$-\mathcal{M}^+(D^2(-u)) + \beta(-u) = f(-x),$$

that leads to that $u^- = \max\{-u, 0\}$ is a subsolution of (1.1) with f^- as right hand side. Therefore we conclude that $|u| = \max\{u^+, u^-\}$ satisfies (2.5). □

By the above lemma we only need to prove a priori estimates of subsolutions of the equation

$$-\mathcal{M}^+(D^2u) + \beta(u) = f. \tag{2.6}$$

Proposition 2.1. *[**A priori estimate for** $f \in L^\infty_{loc}(\Omega)$] Let $f \in L^\infty_{loc}(\Omega)$, $f \ge 0$. If β satisfies (β_0) and (β_1), and $u \in \mathcal{C}(\Omega)$ is a nonnegative $\mathcal{C}$-viscosity subsolution of (2.6) in Ω, then there exists R_0 such that for all $0 < R' < R_0$ such that $B_{R'}(z) \subset \Omega$ and for all $0 < R < R'$, the following estimate holds*

$$\sup_{B_R(z)} u \le C\big(1 + \|f\|_{L^\infty(B_{R'}(z))}\big),$$

where $C = C(\beta, R, R', N, \lambda, \Lambda)$ does not depend on f.

Proof. Recall that $\mathbf{B}(t) = \int_0^t \beta(s)\,ds$. Hence, from (β_1) and the fundamental calculus theorem one has $\mathbf{B}(t) \le t\beta(t)$ for all $t > 0$. Then,

$$0 \le \frac{\mathbf{B}(t)}{\beta(t)^2} \le \frac{t}{\beta(t)}.$$

In this way, taking $t \to \infty$ on the above inequality, from (2.4) we obtain

$$\lim_{t\to+\infty} \frac{\sqrt{\mathbf{B}(t)}}{\beta(t)} = 0. \tag{2.7}$$

Now, let $\hat{\phi}$ be the unique solution of the problem

$$\begin{cases} |\hat{\phi}'(\delta)|^2 = \dfrac{1}{2}\mathbf{B}(\hat{\phi}(\delta)), & \delta > 0, \\ \hat{\phi}(\delta) \to +\infty \text{ as } \delta \to 0^+. \end{cases}$$

Note that $\hat{\phi}'(\delta) = -(1/\sqrt{2})\sqrt{\mathbf{B}(\hat{\phi}(\delta))} < 0$ for all $\delta > 0$, and $\hat{\phi}''(\delta) = (1/4)\beta(\hat{\phi}(\delta)) > 0$ for all $\delta > 0$. Therefore, using the change of variable $t = \hat{\phi}(\delta)$ in (2.7) and the fact that $\hat{\phi}(\delta) \to +\infty$ as $\delta \to 0^+$, we obtain

$$\lim_{\delta\to0^+} \frac{\hat{\phi}'(\delta)}{\hat{\phi}''(\delta)} = -2\sqrt{2} \lim_{\delta\to0^+} \frac{\sqrt{\mathbf{B}(\hat{\phi}(\delta))}}{\beta(\hat{\phi}(\delta))} = 0. \tag{2.8}$$

Now, for $\rho > 0$ to be fixed later, we define

$$\Phi(x) = \hat{\phi}\left((\rho R')^{-1}\left((R')^2 - |x-z|^2\right)\right), \quad x \in B_{R'}(z).$$

Then, putting $r = |x-z|$ we obtain

$$\nabla\Phi(x) = -2(\rho R')^{-1}\hat{\phi}'\left((\rho R')^{-1}((R')^2 - r^2)\right)(x-z)$$

and

$$D^2\Phi(x) = -2(\rho R')^{-1}\hat{\phi}'\left((\rho R')^{-1}((R')^2 - r^2)\right) I + 4(\rho R')^{-2}\hat{\phi}''\left((\rho R')^{-1}((R')^2 - r^2)\right) X,$$

for all $x \in B_{R'}(z)$, where I and X are matrices of order $N \times N$, being I the identity matrix and $X = ((x_i - z_i)(x_j - z_j))_{i,j=1}^N$. Hence, for every vector ξ such that $(\xi - z)\cdot(x-z) = 0$ one has

$$D^2\Phi(x)(\xi - z) = -2(\rho R')^{-1}\hat{\phi}'\left((\rho R')^{-1}((R')^2 - r^2)\right)(\xi - z),$$

and on the other hand, also one has

$$D^2\Phi(x)(x-z) = \Big(-2(\rho R')^{-1}\hat{\phi}'\left((\rho R')^{-1}((R')^2 - r^2)\right) + 4(\rho R')^{-2}r^2\hat{\phi}''\left((\rho R')^{-1}((R')^2 - r^2)\right)\Big)(x-z)$$

for all $x \in B_{R'}(z)$. Therefore for $D^2\Phi$, the Hessian matrix of Φ, the eigenvalues associates are

$$-2(\rho R')^{-1}\hat{\phi}'\left((\rho R')^{-1}((R')^2 - r^2)\right),$$

which has multiplicity $N-1$ and that is strictly positive for all $R' > 0$ sufficiently small according to (2.8), and

$$-2(\rho R')^{-1}\hat{\phi}'\left((\rho R')^{-1}((R')^2 - r^2)\right) + 4(\rho R')^{-2}r^2\hat{\phi}''\left((\rho R')^{-1}((R')^2 - r^2)\right),$$

which is simple and strictly positive for any R'. It follows that

$$\begin{aligned} &-\mathcal{M}^+(D^2\Phi(x)) + \frac{1}{2}\beta(\Phi(x)) \\ &= 2\Lambda(\rho R')^{-1}\hat{\phi}''\left((\rho R')^{-1}((R')^2 - r^2)\right)\left(N\frac{\hat{\phi}'\left((\rho R')^{-1}((R')^2 - r^2)\right)}{\hat{\phi}''\left((\rho R')^{-1}((R')^2 - r^2)\right)} - 2r^2(\rho R')^{-1} + \frac{1}{4\Lambda}\right), \end{aligned}$$

for all $x \in B_{R'}(z)$. Now, note that $\rho \geq 1$ implies that

$$(\rho R')^{-1}((R')^2 - r^2) = \frac{(R')^2 - r^2}{\rho R'} \leq \frac{1}{\rho}\frac{(R')^2}{R'} \leq \frac{R'}{\rho} \leq R' \quad \text{for all } 0 < r < R'.$$

On the other hand, from (2.8), there exists $R_0 > 0$ sufficiently small such that if $0 < R' < R_0$, then

$$0 < -\frac{\hat{\phi}'\left((\rho R')^{-1}((R')^2 - r^2)\right)}{\hat{\phi}''\left((\rho R')^{-1}((R')^2 - r^2)\right)} < \frac{1}{8N\Lambda},$$

and if we choose $\rho = 8\Lambda > 1$ and $R_0 < 1$ sufficiently small, we obtain,

$$0 < 2r^2(\rho R')^{-1} = \frac{2r^2}{\rho R'} < \frac{2(R')^2}{\rho R'} = \frac{2R'}{\rho} < \frac{1}{\rho} < \frac{1}{8\Lambda}.$$

Therefore

$$-\mathcal{M}^+(D^2\Phi) + \frac{1}{2}\beta(\Phi) > 0 \quad \text{in } B_{R'}(z).$$

Observe that (2.4) implies that

$$\beta(2\|f\|_{L^\infty(B_{R'}(z))} + C_1) \geq 2\|f\|_{L^\infty(B_{R'}(z))} + C_1, \quad \text{for some } C_1 \geq 0 \text{ sufficiently large.}$$

In this way, we can choose and fix some inverse image of $2\|f\|_{L^\infty(B_{R'}(z))} + C_1$ that verifies

$$\beta^{-1}\big(2\|f\|_{L^\infty(B_{R'}(z))} + C_1\big) \leq 2\|f\|_{L^\infty(B_{R'}(z))} + C_1,$$

and consider the function

$$\Psi(x) = \Phi(x) + \beta^{-1}\left(2\|f\|_{L^\infty(B_{R'}(z))} + C_1\right), \quad x \in B_{R'}(z).$$

Since
$$\beta\big(\Phi(x)+\beta^{-1}\big(2\|f\|_{L^\infty(B_{R'}(z))}+C_1\big)\big)\geq\beta\big(\beta^{-1}\big(2\|f\|_{L^\infty(B_{R'}(z))}+C_1\big)\big)$$
and
$$\beta(\Phi(x))\leq\beta\big(\Phi(x)+\beta^{-1}\big(2\|f\|_{L^\infty(B_{R'}(z))}+C_1\big)\big),$$
for all $x\in B_{R'}(z)$, we obtain
$$\begin{aligned}&\beta\big(\Phi(x)+\beta^{-1}\big(2\|f\|_{L^\infty(B_{R'}(z))}+C_1\big)\big)-\beta(\Phi(x))\\&\qquad\geq\beta\big(\beta^{-1}\big(2\|f\|_{L^\infty(B_{R'}(z))}+C_1\big)\big)-\beta\big(\Phi(x)+\beta^{-1}\big(2\|f\|_{L^\infty(B_{R'}(z))}+C_1\big)\big),\end{aligned}$$
for all $x\in B_{R'}(z)$. Hence
$$\beta\big(\Phi(x)+\beta^{-1}\big(2\|f\|_{L^\infty(B_{R'}(z))}+C_1\big)\big)\geq\frac{1}{2}\big(\beta(\Phi(x))+2\|f\|_{L^\infty(B_{R'}(z))}+C_1\big),$$
for all $x\in B_{R'}(z)$. In this way, from properties of Φ, we have
$$\begin{aligned}-\mathcal{M}^+(D^2\Psi(x))+\beta(\Psi(x))-f(x)\ &\geq-\mathcal{M}^+(D^2\Phi(x))+\frac{1}{2}\beta(\Phi(x))\\&\quad+\|f\|_{L^\infty(B_{R'}(z))}+\frac{C_1}{2}-f(x)\\&\geq\|f\|_{L^\infty(B_{R'}(z))}+\frac{C_1}{2}-f(x)\\&\geq 0,\quad\text{for all } x\in B_{R'}(z).\end{aligned}$$
Thus $\Psi\in\mathcal{C}^2(B_{R'}(z))$ is a strong positive supersolution of (2.6) in $B_{R'}(z)$, such that
$$\lim_{|x-z|\to R'}\Psi(x)=\infty.$$
On the other hand, $u\in\mathcal{C}(\Omega)$ is a nonnegative $\mathcal{C}$-viscosity subsolution of (2.6) in Ω, therefore bounded in $B_{R'}(z)$. From Lemma 2.1, it follows that $u<\Psi$ in $B_{R'}(z)$.

Finally, since
$$\Phi(x)=\hat\phi\big((\rho R')^{-1}((R')^2-|x-z|^2)\big)<C_2=C_2(\beta,R,R',N,\lambda,\Lambda),\quad\text{for all } x\in B_{R'}(z),$$
and $\beta^{-1}\big(2\|f\|_{L^\infty(B_{R'}(z))}+C_1\big)\leq 2\|f\|_{L^\infty(B_{R'}(z))}+C_1$, the result follows. □

Proposition 2.2. *[**A priori estimate for** $f\in L^N_{loc}(\Omega)$] Let $f\in L^N_{loc}(\Omega)$, $f\geq 0$. Assume that β satisfies (β_0), (β_1), (β_2) and (β_3). If $u\in\mathcal{C}(\Omega)$ is a nonnegative $\mathcal{C}$-viscosity subsolution of (2.6), then there exists R_0 such that for all $0<R'<R_0$ such that $B_{R'}(z)\subset\Omega$ and for all $0<R<R'$, the following estimate holds*
$$\sup_{B_R(z)}u\leq C(1+\|f\|_{L^N(B_{R'}(z))}),$$
where $C=C(\beta,R,R',N,\lambda,\Lambda)$ does not depend on f.

Proof. Let $v=\frac{1}{\phi(\xi)}u$, with $\xi=(R')^2-|x-z|^2$, and let $\phi=\psi^{-1}$, with ψ^{-1} defined as (1.3), the solution of the problem
$$\begin{cases}\phi''(\delta)=\beta(\phi(\delta)), & \delta>0,\\ \phi(\delta)\to+\infty \text{ as } \delta\to 0^+.\end{cases}$$
We want to find the equation that v satisfies. Suppose that $v-\varphi$ has a local maximum, $v(\hat x)-\varphi(\hat x)$, $Dv(\hat x)=D\varphi(\hat x)$ and $\varphi\in\mathcal{C}^2(\Omega)$. Then $u-\phi(\xi)\varphi$ has a local maximum at $\hat x$. Therefore $\phi(\xi)\varphi$ is a test function for u and so
$$-\frac{1}{\phi(\xi)}\mathcal{M}^+(D^2(\phi(\xi)\varphi))+\frac{1}{\phi(\xi)}\beta(\phi(\xi)\varphi)\leq\frac{f}{\phi(\xi)}$$

or equivalently

$$-\frac{1}{\phi(\xi)}\mathcal{M}^+\left(\varphi\phi''(\xi)(D\xi\otimes D\xi)+\phi'(\xi)\varphi D^2\xi+\phi'(\xi)(D\xi\otimes D\varphi)+\phi'(\xi)(D\varphi\otimes D\xi)+\phi(\xi)D^2\varphi\right)$$
$$+\frac{1}{\phi(\xi)}\beta(\phi(\xi)\varphi)\le\frac{f}{\phi(\xi)}.$$

Now we replace φ by v and $D\varphi$ by Dv for obtaining

$$-\mathcal{M}^+\left(D^2v\right)-v\frac{\phi''(\xi)}{\phi(\xi)}\mathcal{M}^+(D\xi\otimes D\xi)+v\frac{\phi'(\xi)}{\phi(\xi)}\mathcal{M}^-(D^2\xi)+\frac{\phi'(\xi)}{\phi(\xi)}\mathcal{M}^-\left(D\xi\otimes Dv+Dv\otimes D\xi\right)$$
$$+\frac{1}{\phi(\xi)}\beta(\phi(\xi)v)\le\frac{f}{\phi(\xi)}$$

in $B_{R'}(z)$ in the $\mathcal{C}$-viscosity sense.

In what follows we write $\Omega^+=\{x\in\Omega: v(x)>0\}$. Consider the contact set for the function v, which is defined as

$$\Gamma_v^+=\{x\in B_{R'}(z):\exists p\in\mathbb{R}^N\text{ with } v(y)\le v(x)+\langle p,y-x\rangle,\forall y\in B_{R'}(z)\},$$

where $R'>0$ is sufficiently small. We observe that $\Gamma_v^+\subset\Omega^+\cap B_{R'}(z)$ and that if $\overline{v}$ is the concave envelope of v in $\overline{B_{R'}(z)}$ then for $x\in B_{R'}(z)$ we have $v(x)=\overline{v}(x)$ if and only if $x\in\Gamma_v^+$. The function $\overline{v}$, being concave, satisfies

$$\overline{v}(y)\le v(x)+\langle Dv(x),y-x\rangle,\quad\forall x\in\Gamma_v^+,\forall y\in\overline{B_{R'}(z)}.$$

Choosing adequately $y\in\partial B_{R'}(z)$ we obtain

$$|Dv(x)|\le\frac{v(x)}{R'-|x-z|}\le\frac{2R'}{\xi}v(x),\quad\forall x\in\Gamma_v^+.$$

On the other hand, by means straightforward calculations, since β satisfies (β_0), (β_1) and (β_3) we obtain

$$\frac{\phi'(\xi)}{\xi\phi''(\xi)}<\tilde{K}\quad\text{for some }\tilde{K}=\tilde{K}(\beta)>0,$$

independently of all R' sufficiently small. In consequence, since $\phi''(\xi)=\beta(\phi(\xi))$, we have that

$$\left|-v\frac{\phi''(\xi)}{\phi(\xi)}\mathcal{M}^+(D\xi\otimes D\xi)+v\frac{\phi'(\xi)}{\phi(\xi)}\mathcal{M}^-(D^2\xi)+\frac{\phi'(\xi)}{\phi(\xi)}\mathcal{M}^-\left(D\xi\otimes Dv+Dv\otimes D\xi\right)\right|<Kv\frac{\beta(\phi(\xi))}{\phi(\xi)},$$

for some constant $K=K(\beta,N,\lambda,\Lambda)>0$. In this way, v satisfies

$$-\mathcal{M}^+(D^2v)+v\frac{\beta(\phi(\xi))}{\phi(\xi)}\left(\frac{\beta(v\phi(\xi))}{v\beta(\phi(\xi))}-K\right)\le\frac{f}{\phi(\xi)},\quad\forall x\in\Gamma_v^+.$$

Let $M>1$ a constant to be fixed later. For $x\in\Gamma_v^+$ such that $v>M$, from assumption (β_3) we get

$$\frac{\beta(v\phi(\xi))}{v^q(\phi(\xi))^q}\ge\frac{\beta(\phi(\xi))}{(\phi(\xi))^q},$$

for some $q>1$. Hence, it follows that

$$\frac{\beta(v\phi(\xi))}{v\beta(\phi(\xi))}\ge v^{q-1}\ge M^{q-1}.$$

Now we choose $M>(\max\{1,K\})^{1/(q-1)}$, and define $w=\max\{v-M,0\}$ in $B_{R'}(z)$. Observe that $\Gamma_w^+\subset\Gamma_v^+$ and $\Gamma_w^+\subset\{x\in B_{R'}(z):w>0\}$. Hence, we obtain

$$-\mathcal{M}^+(D^2w)\le\frac{f}{\phi(\xi)}\quad\text{a.e. in }\Gamma_w^+,$$

and from the Alexandroff-Bakelman-Pucci inequality it follows that

$$\sup_{B_{R'}(z)} w \le \tilde{C} \left\| \frac{f}{\phi(\xi)} \right\|_{L^N(B_{R'}(z))},$$

for some constant $\tilde{C} = \tilde{C}(\beta, R', N, \lambda, \Lambda) > 0$. Then,

$$C_1 \sup_{B_R(z)} u \le \sup_{B_{R'}(z)} v \le \sup_{B_{R'}(z)} w + M \le C_2 \left(1 + \|f\|_{L^N(B_{R'}(z))}\right),$$

where C_1 and C_2 are constants depending only on β, R, R', N, λ and Λ, but independents on f. □

Proof of Theorem 1.1 and Theorem 1.2. In case $f \in L^\infty_{loc}(\Omega)$, since there exists $g \in L^\infty(\Omega)$ such that $f \ge g$, we can find an increasing sequence of continuous functions $\{f_n\}_n \subset L^\infty(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that

$$\lim_{n\to\infty} \|f_n - f\|_{L^\infty(\Omega')} = 0 \quad \forall \Omega' \subset\subset \Omega.$$

In case $f \in L^N_{loc}(\Omega)$ we have a similar situation. Since there exists $g \in L^N(\Omega)$ such that $f \ge g$, we can find an increasing sequence of continuous functions $\{f_n\}_n \subset L^N(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that

$$\lim_{n\to\infty} \int_{\Omega'} |f_n - f|^N = 0 \quad \forall \Omega' \subset\subset \Omega.$$

Then in booth cases from Lemma 2.2 we can find $u_n \in \mathcal{C}^{1,\alpha}(\Omega)$ being a $\mathcal{C}$-viscosity solution to problem (2.1) with f_n as a right hand side. By Lemma 2.1 we obtain that $u_n \le u_{n+1}$ in Ω.

According to Lemma 2.3, Proposition 2.1 or Proposition 2.2, for every $\Omega' \subset\subset \Omega$ we have

$$\sup_{\Omega'} |u_n| \le C,$$

where C is a constant that does not depend on n. Observe that in case $f \ge 0$ we obtain directly the conclusion without using the Lemma 2.3, because in this case $u_n \ge 0$. Using now the Proposition 4.2 in [8] and a diagonal argument, there exists a subsequence that converges uniformly in compact set to u. Moreover, $u \ge u_n$ in Ω, for all n, so that $\liminf_{x\to\partial\Omega} u \ge n$, for all $n \in \mathbb{N}$. Finally, using the Proposition 3.8 in [7] we can conclude that u is an L^∞-viscosity (respectively, L^N-viscosity) large solution of (1.1). □

3. Uniqueness and asymptotic behavior near the boundary of large solutions

Let Ω be an open bounded domain of $\mathbb{R}^N$, $N > 1$, with $\partial\Omega$ of $\mathcal{C}^2$ class, and let us assume that $\beta \in \mathcal{C}(0, \infty)$ is a positive continuous nondecreasing function such that it verifies the Keller-Osserman condition. We remark that in all this section we assume that $f \in \mathcal{C}(\Omega)$, therefore our results are related with $\mathcal{C}$-viscosity large solutions of (1.1).

We start mention a well known result related with the distance function, which will be of utility in our approach.

Lemma 3.1. *Let Ω be a bounded domain of $\mathbb{R}^N$, $N > 1$, with $\partial\Omega$ nonempty. Then $\delta(\cdot)$ is a Lipschitz continuous function in $\mathbb{R}^N$. If in addition we assume that $\partial\Omega$ is of class $\mathcal{C}^k$, $k \ge 2$, then there exists a constant $\mu_\Omega > 0$ such that*

$$\delta(\cdot) \in \mathcal{C}^k(\Gamma_{\mu_\Omega}),$$

where $\Gamma_{\mu_\Omega} = \{x \in \Omega : 0 < \delta(x) < \mu_\Omega\}$. Moreover, if $x \in \Gamma_{\mu_\Omega}$ and $\bar{x} = \bar{x}(x)$ is the only one point on $\partial\Omega$ such that $\delta(x) = |x - \bar{x}|$, then, in terms of a principal coordinate system at $\bar{x}$, we have that

$$(D\delta(x) \otimes D\delta(x)) = \operatorname{diag}[0, \ldots, 0, 1]$$

and

$$D^2\delta(x) = \operatorname{diag}\left[\frac{-\kappa_1}{1 - \kappa_1\delta(x)}, \ldots, \frac{-\kappa_{N-1}}{1 - \kappa_{N-1}\delta(x)}, 0\right],$$

where κ_i are the principal curvatures of $\partial\Omega$ at $\bar{x}$.

The proof of the previous lemma may be found in [16].

If (β_4) is assumed, one another result of utility for our proof is the following

Lemma 3.2. *Assume that ψ is strictly monotone decreasing and satisfies (β_4). Then for every $\gamma > 1$ there exist positive numbers η_γ, δ_γ, such that*

$$\phi((1-\eta)\delta) \leq \gamma\phi(\delta), \quad \text{for all } \eta \in [0,\eta_\gamma], \quad \text{for all } \delta \in [0,\delta_\gamma].$$

This lemma is the Lemma C in [3]. One proof can be found in [15].

In the proof of every one of the two propositions below, we follow the lineaments used in [21, 11], whose works were considered cases involving the function $\beta(t) = t^p$, $p > 1$. In particular, we consider the situation for f associated to the Theorem 1.3. This is, f is a continuous function in Ω that verifies (1.4).

We start obtaining upper estimates near the boundary of local solutions of (1.1) which can be derived from the following proposition.

Proposition 3.1. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N > 1$, with $\partial\Omega$ of class $\mathcal{C}^2$, let $f \in \mathcal{C}(\Omega)$ such that*

$$\limsup_{\delta(x)\to 0} \frac{f(x)}{\beta\big(\phi\big(\sqrt{(1-\eta)F(A)^{-1}}(\delta(x)-\mu)\big)\big)} \leq \eta \tag{3.1}$$

holds and let $\eta \in [0,1[$. If β satisfies (β_0), (β_1) and (β_4), then for every nonnegative $\mathcal{C}$-viscosity subsolution u of (1.1) one has

$$\limsup_{\delta(x)\to 0} \frac{u(x)}{\phi\big(\sqrt{(1-\eta)F(A)^{-1}}\delta(x)\big)} \leq 1. \tag{3.2}$$

Proof. Let $\mu \in\,]0,\mu_1[$, with $0 < \mu_1 < \mu_\Omega$ to be fixed later, $0 < \tau < 1-\eta$ and $K_1 > 0$ to be chosen later, and let us consider in $\Omega_{\mu,\mu_1} = \{x \in \Omega : \mu < \delta(x) < \mu_1\}$ the function

$$\Psi_\tau^-(x) = \phi\big(\sqrt{\tau F(A)^{-1}}(\delta(x)-\mu)\big) + K_1.$$

Hence,

$$\begin{aligned}
&F\left(D^2\Psi_\tau^-(x)\right) \\
&\leq F\big(\tau F(A)^{-1}\phi''\big(\sqrt{\tau F(A)^{-1}}(\delta(x)-\mu)\big)\nabla\delta(x)\otimes\nabla\delta(x)\big) \\
&\quad +\mathcal{M}^+\big(\sqrt{\tau F(A)^{-1}}\phi'\big(\sqrt{\tau F(A)^{-1}}(\delta(x)-\mu)\big)D^2\delta(x)\big) \\
&= \tau\phi''\big(\sqrt{\tau F(A)^{-1}}(\delta(x)-\mu)\big) + \sqrt{\tau F(A)^{-1}}\phi'\big(\sqrt{\tau F(A)^{-1}}(\delta(x)-\mu)\big)\mathcal{M}^-(D^2\delta(x))
\end{aligned}$$

in Ω_{μ,μ_1}. Since

$$\lim_{\delta(x)\to\mu^+} \frac{\phi'\big(\sqrt{\tau F(A)^{-1}}(\delta(x)-\mu)\big)}{\phi''\big(\sqrt{\tau F(A)^{-1}}(\delta(x)-\mu)\big)} = \lim_{\delta(x)\to\mu^+} \frac{\phi'\big(\sqrt{\tau F(A)^{-1}}(\delta(x)-\mu)\big)}{\beta(\Psi_\tau^-(x)-K_1)} = 0,$$

and (3.1) holds, from Lemma 3.1 one has

$$\begin{aligned}
&-F\big(D^2\Psi_\tau^-(x)\big) + \beta\big(\Psi_\tau^-(x)\big) - f(x) \\
&\quad\geq\ \beta\big(\phi\big(\sqrt{\tau F(A)^{-1}}(\delta(x)-\mu)\big)\big) \\
&\qquad\times\Big((1-\eta-\tau) - \varepsilon - \frac{\sqrt{\tau F(A)^{-1}}\phi'\big(\sqrt{\tau F(A)^{-1}}(\delta(x)-\mu)\big)\mathcal{M}^-\big(D^2\delta(x)\big)}{\phi''\big(\sqrt{\tau F(A)^{-1}}(\delta(x)-\mu)\big)}\Big)
\end{aligned}$$

in Ω_{μ,μ_1} with μ_1 sufficiently small. Hence, bearing in mind that $\tau < 1-\eta$, we can choose $\varepsilon < 1-\eta-\tau$ and μ_1 sufficiently small in order to conclude that

$$-F\big(D^2\Psi_\tau^-(x)\big) + \beta\big(\Psi_\tau^-(x)\big) \geq f(x) \quad \forall x \in \Omega_{\mu,\mu_1}.$$

Choosing now $K_1 = K_1(\mu_1) = \max\{u(y) : \delta(y) \geq \mu_1\}$, and comparing in Ω_{μ,μ_1}, we get

$$u(x) \leq \Psi_\tau^-(x), \quad \forall x \in \Omega_{\mu,\mu_1}.$$

In this way, we obtain

$$\frac{u(x)}{\phi\big(\sqrt{(1-\eta)F(A)^{-1}}(\delta(x)-\mu)\big)} \leq \frac{\phi\big(\sqrt{\tau F(A)^{-1}}(\delta(x)-\mu)\big)+K_1}{\phi\big(\sqrt{(1-\eta)F(A)^{-1}}(\delta(x)-\mu)\big)} \quad \text{in } \Omega_{\mu,\mu_1}.$$

Therefore, making $\mu \searrow 0$ and later $\tau \nearrow (1-\eta)$, we conclude that (3.2) holds. □

Next result deals to obtain lower estimates near the boundary of local solutions of (1.1)-(1.2) with $f \in \mathcal{C}(\Omega)$.

Proposition 3.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N > 1$, with $\partial\Omega$ of class $\mathcal{C}^2$, let $f \in \mathcal{C}(\Omega)$, $f \geq 0$, such that*

$$\liminf_{\delta(x)\to 0} \frac{f(x)}{\beta\big(\phi\big(\sqrt{(1-\eta)F(A)^{-1}}(\delta(x)-\mu)\big)\big)} \geq \eta$$

holds and let β a function that satisfies (β_0), (β_1) and (β_4). Then, for every nonnegative $\mathcal{C}$-viscosity large solution u of (1.1) one has

$$\liminf_{\delta(x)\to 0} \frac{u(x)}{\phi\big(\sqrt{(1-\eta)F(A)^{-1}}\delta(x)\big)} \geq 1. \tag{3.3}$$

Proof. Let $\mu \in]0,\mu_2[$, with $0 < \mu_2 < \mu_\Omega/2$ a constant to be fixed later and we define in $\Omega_{0,\mu_2} = \{x \in \Omega : 0 < \delta(x) < \mu_2\}$ the function

$$\Psi_\tau^+(x) = \phi\big(\sqrt{\tau F(A)^{-1}}(\delta(x)+\mu)\big) - \phi\big(\sqrt{\tau F(A)^{-1}}(\mu_1+\mu)\big).$$

Similarly to the proof of Proposition 3.1, for μ_2 sufficiently small and $\tau > 1-\eta$ one can obtain

$$u(x) \geq \Psi_\tau^+(x), \quad \forall x \in \Omega_{\mu,\mu_1}.$$

where $\Omega_{0,\mu_2} = \{x \in \Omega : 0 < \delta(x) < \mu_2 < \min\{1/2, \mu_\Omega/2\}\}$. Then, after dividing the above inequality by $\phi\big(\sqrt{(1-\eta)F(A)^{-1}}(\delta(x)+\mu)\big)$, making $\mu \searrow 0$ and later $\tau \searrow (1-\eta)$, we conclude that (3.3) holds. □

Now we have all ingredients in order to prove the Theorem 1.3.

Proof of the Theorem 1.3. Since $f \in \mathcal{C}(\Omega)$, $f \geq 0$, such that (1.4) holds, from (3.2) and (3.3), one has that every nonnegative $\mathcal{C}$-viscosity large solution w of (1.1) verifies

$$\lim_{\delta(x)\to 0} \frac{w(x)}{\phi\big(\sqrt{(1-\eta)F(A)^{-1}}\delta(x)\big)} = 1.$$

For uniqueness, note that if u and v are two $\mathcal{C}$-viscosity large solutions of (1.1), then

$$\lim_{\delta(x)\to 0} \frac{u(x)}{v(x)} = 1.$$

In particular this implies that for $\varepsilon > 0$ given one has

$$\lim_{\delta(x)\to 0} \frac{u(x)}{(1+\varepsilon)v(x)} = \frac{1}{1+\varepsilon} < 1.$$

Besides, since from (β_5) and the fact that $v \in \mathcal{C}(\Omega)$ is nonnegative, we have that $\beta((1+\varepsilon)v) \geq (1+\varepsilon)\beta(v)$, and since $f \geq 0$ we get

$$-F(D^2(1+\varepsilon)v) + \beta((1+\varepsilon)v) \geq -(1+\varepsilon)F(D^2v) + (1+\varepsilon)\beta(v) = (1+\varepsilon)f \geq f \quad \text{in } \Omega.$$

Hence,

$$-F(D^2u) + \beta(u) \leq -F(D^2(1+\varepsilon)v) + \beta((1+\varepsilon)v) \quad \text{in } \Omega.$$

It follows from Lemma 2.1 that $u \le (1+\varepsilon)v$ in Ω. Now, passing to the limit as $\varepsilon \to 0$ we obtain $u \le v$ in Ω. Interchanging roles between u and v, we also obtain that $u \ge v$ in Ω. Therefore,

$$u = v \quad \text{in } \Omega.$$

□

Our next two results show uniqueness and behavior asymptotic near the boundary of solutions of the problem (1.1)-(1.2) when $f(u) = u^p$ or $f(u) = e^u$, extending the examples given in the Introduction. Before, note that for every $C \ge 0$ there exists a unique $\eta = \eta(C) \in [0, 1[$ such that

$$C = \frac{\eta}{1-\eta}.$$

Corollary 3.1. *Assume $p > 1$ and $0 < \alpha \le 2p/(p-1)$. Let Ω be a bounded open subset of $\mathbb{R}^N$, $N > 1$, with $\partial\Omega$ of class $\mathcal{C}^2$, and let $f \in \mathcal{C}(\Omega)$, $f \ge 0$, such that*

$$\limsup_{\delta(x)\to 0} f(x)(\delta(x))^\alpha \le C,$$

holds, for some $C \ge 0$. Then the equation (1.1), with $f(t) = t^p$ has a unique nonnegative $\mathcal{C}$-viscosity large solution $u \in \mathcal{C}(\Omega)$. Moreover,

$$\lim_{\delta(x)\to 0} u(x)\left(\left(\frac{2(p+1)(1-\eta)}{(p-1)^2 F(A)}\right)^{\frac{1}{p-1}} (\delta(x))^{\frac{2}{p-1}}\right) = 1,$$

where $\eta = \eta(C)$.

Corollary 3.2. *Assume $0 < \alpha \le 2$. Let Ω be a bounded open subset of $\mathbb{R}^N$, $N > 1$, with $\partial\Omega$ of class $\mathcal{C}^2$ and let $f \in \mathcal{C}(\Omega)$, $f \ge 0$, such that*

$$\limsup_{\delta(x)\to 0} f(x)(\delta(x))^\alpha \le C$$

holds, for some $C \ge 0$. Then the equation (1.1), with $f(t) = e^t$ has a unique nonnegative $\mathcal{C}$-viscosity large solution $u \in \mathcal{C}(\Omega)$. Moreover,

$$\lim_{\delta(x)\to 0} u(x)\left(\log \frac{2F(A)}{(1-\eta)(\delta(x))^2}\right)^{-1} = 1,$$

where $\eta = \eta(C)$.

Finally, for the particular choice $\beta(t) = t^p$, $p > 1$, we finish this section showing an example in which the condition (1.4) does not hold. In this situation, it is convenient to introduce the following function

$$\widehat{v}(\delta) = \tau\delta^{-\alpha}, \quad \delta > 0,$$

where $\alpha > 0$ and $\tau > 0$ are given.

Theorem 3.1. *Assume $p > 1$ and $\alpha > \frac{2}{p-1}$. Let Ω be a bounded open subset of $\mathbb{R}^N$, $N > 1$, with $\partial\Omega$ of class $\mathcal{C}^2$ and $f \in \mathcal{C}(\Omega)$, $f \ge 0$, such that*

$$\lim_{\delta(x)\to 0} f(x)(\delta(x))^{\alpha p} = C^p, \tag{3.4}$$

holds, for some constant $C > 0$. Then the equation (1.1) with $\beta(t) = t^p$ has a unique nonnegative $\mathcal{C}$-viscosity large solution $u \in \mathcal{C}(\Omega)$. Moreover,

$$\lim_{\delta(x)\to 0} u(x)(\delta(x))^\alpha = C. \tag{3.5}$$

Proof. Let $\mu \in]0, \mu_1[$, with $0 < \mu_1 < \mu_\Omega$ to be fixed later, $\tau > C$ and $K_1 > 0$ to be chosen later, and let us consider $\Omega_{\mu,\mu_1} = \{x \in \Omega : \mu < \delta(x) < \mu_1\}$ the function

$$\Psi_\tau^-(x) = \widehat{v}\,(\delta(x) - \mu) + K_1.$$

Hence, from Lemma 3.1, straightforward calculations lead to

$$\begin{aligned}&-F\big(D^2\Psi_\tau^-(x)\big) + (\Psi_\tau^-(x))^p \\ &\geq (\delta(x)-\mu)^{-\alpha p}\left(\tau^p - \alpha(\alpha+1)F(A)\tau\,(\delta(x)-\mu)^{\theta_1} + \alpha F(A)\tau(\delta(x)-\mu)^{\theta_1+1}\left\|D^2\delta\right\|_\infty\right)\end{aligned}$$

in Ω_{μ,μ_1}, for some $\theta_1 > 0$ such that $\alpha p = \alpha + 2 + \theta_1$. On the other hand, from (3.4), for every $\varepsilon > 0$ one has

$$0 \leq f(x)(\delta(x))^{\alpha p} \leq C^p + \varepsilon \quad \text{if } 0 < \delta(x) < \mu_1,$$

for some $\mu_1^* \in]0, \mu_\Omega[$, which implies that

$$-f(x) \geq -(C^p + \varepsilon)(\delta(x) - \mu)^{\alpha p} \quad \text{if } 0 < \delta(x) < \mu_1^*.$$

Then, for $\mu < \delta(x) < \mu_1 < \mu_\Omega$ and choosing $C < \tau < 2C$, it follows that

$$-F(D^2\Psi_\tau^-(x)) + (\Psi_\tau^-(x))^p - f(x) \geq (\delta(x)-\mu)^{-\alpha p}\left(\tau^p - C^p - \varepsilon - \alpha(\alpha+1)F(A)(2C)\mu_1^{\theta_1}\right).$$

Hence, bearing in mind that $\tau > C$, we can choose $\varepsilon < \tau^p - C^p$ and $0 < \mu_1 < \mu_\Omega$ sufficiently small in order to conclude that

$$-F(D^2\Psi_\tau^-(x)) + (\Psi_\tau^-(x))^p \geq f(x) \quad \text{if } \mu < \delta(x) < \mu_1.$$

Considering now $K = K(\mu_1) = \max\{u(y) : \delta(y) \geq \mu_1\}$, and comparing in the region in Ω_{μ,μ_1}, we get

$$u(x) \leq \Psi_\tau^-(x), \quad \forall x \in \Omega_{\mu,\mu_1}.$$

In this way, it follows that

$$u(x)\,(\delta(x)-\mu)^\alpha \leq \tau + K(\mu_1)\,(\delta(x)-\mu)^\alpha \quad \text{if } \mu < \delta(x) < \mu_1.$$

Therefore, making $\mu \searrow 0$ and later $\tau \searrow C$, we obtain

$$\limsup_{\delta(x)\to 0} u(x)(\delta(x))^\alpha \leq C. \tag{3.6}$$

Arguing similarly for $\tau < C$, we obtain that

$$u(x) \geq \Psi_\tau^+(x) = \widehat{v}\,(\delta(x)+\mu) - \widehat{v}\,(\mu_2+\mu)\,, \quad \forall x \in \Omega_{\mu,\mu_1}.$$

where $\Omega_{\mu,\mu_2} = \{x \in \Omega : \mu < \delta(x) < \mu_2 < \min\{1/2, \mu_\Omega\}\}$, with μ_2 sufficiently small. Then, after multiplying the above inequality by $(\delta(x)-\mu)^\alpha$, making $\mu \searrow 0$ and later $\tau \nearrow C$, we conclude that

$$\liminf_{\delta(x)\to 0} u(x)\,(\delta(x))^\alpha \geq C. \tag{3.7}$$

Combining (3.6) and (3.7) we really obtain (3.5). □

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