

A PANEITZ-TYPE PROBLEM IN PIERCED DOMAINS

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ABSTRACT. We study the critical problem

$$(P_\varepsilon) \quad \begin{cases} \Delta^2 u = u^{\frac{N+4}{N-4}} & \text{in } \Omega \setminus \overline{B(\xi_0, \varepsilon)}, \\ u > 0 & \text{in } \Omega \setminus \overline{B(\xi_0, \varepsilon)}, \\ u = \Delta u = 0 & \text{on } \partial(\Omega \setminus \overline{B(\xi_0, \varepsilon)}), \end{cases}$$

where Ω is an open bounded domain in \mathbb{R}^N , $N \geq 5$, $\xi_0 \in \Omega$ and $B(\xi_0, \varepsilon)$ is the ball centered at ξ_0 with radius $\varepsilon > 0$ small enough. We construct solutions of (P_ε) blowing-up at the center of the hole as the size of the hole goes to zero.

1. INTRODUCTION

This paper deals with the following fourth order problem involving the bi-Laplacian operator

$$(1.1) \quad \begin{cases} \Delta^2 u = u^{\frac{N+4}{N-4}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded domain in \mathbb{R}^N and $N \geq 5$. The exponent $\frac{N+4}{N-4}$ is the critical Sobolev exponent for the embedding $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$. The interest in this equation grew up from its resemblance to some geometric equations involving Paneitz operator and widely studied in the last years by Branson-Chang-Yang [2], Chang [3], Chang-Gurski-Yang [4] and Chang-Yang [5]. Solvability of (1.1) is a delicate issue and depends strongly on the geometry of the domain Ω . Indeed, Van der Vorst [22] proved that (1.1) does not admit any positive solutions when Ω is star-shaped while Ebobisse and Ahmedou [7] showed that (1.1) possesses a solution provided that some homology group of Ω is nontrivial. This topological assumption is sufficient but not necessary, as Gazzola, Grunau and Squassina [9] pointed out, by showing examples of contractible domains on which a solution to (1.1) exists.

In this paper, we are concerned with the case when the domain has a circular hole which shrinks to a point, i.e.

$$(1.2) \quad \begin{cases} \Delta^2 u = u^{\frac{N+4}{N-4}} & \text{in } \Omega_\varepsilon := \Omega \setminus \overline{B(\xi_0, \varepsilon)}, \\ u > 0 & \text{in } \Omega_\varepsilon, \\ u = \Delta u = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where $\xi_0 \in \Omega$ and ε is a small positive parameter. The domain Ω_ε is topologically non trivial, so Ebobisse and Ahmedou [7] ensures the existence of a solution for any $\varepsilon < \text{dist}(\xi_0, \partial\Omega)$. A natural question arises: *which is the asymptotic profile of this solution when the hole shrinks to a point, i.e. ε goes to zero?* Unfortunately, the approach used by Ebobisse and Ahmedou does not allow to get any information about the qualitative properties of the solution they found. That is why we are interested in building a solution to problem (1.2) using a perturbative approach, which naturally provides extremely accurate information about the asymptotic behavior of the solution as the size of the hole converges to zero. Moreover, the main important feature of our point of view is that it is the starting point to

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obtain a large number of positive or sign changing solutions to (1.2) when the domain has one or more small circular holes.

In order to state our main result it is necessary to introduce the bubble, which is the key ingredient of our proof. A bubble is the function $U_{\mu,\xi}$ defined by

$$U_{\mu,\xi}(x) = \alpha_N \left(\frac{\mu}{\mu^2 + |x - \xi|^2} \right)^{\frac{N-4}{2}}, \quad x, \xi \in \mathbb{R}^N, \quad \mu \in \mathbb{R}^+$$

where $\alpha_N := (N(N-4)(N-2)(N+2))^{\frac{N-4}{8}}$. It is well known (see Lin [11]) that they are all the positive solutions in $\mathcal{D}^{2,2}(\mathbb{R}^N)$ of the limit problem

$$(1.3) \quad \Delta^2 U = U^p \quad \text{in } \mathbb{R}^N.$$

To find a good approximation of the solution we are looking for, we need to project the bubble onto the domain Ω_ε with Dirichlet boundary conditions. For any function $u \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ we denote by Pu its protection on $H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon)$, i.e. the unique solution of the problem

$$\begin{cases} \Delta^2 Pu = \Delta^2 u & \text{in } \Omega_\varepsilon, \\ Pu = \Delta Pu = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Our main result reads as follows.

Theorem 1.1. *There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there exists a solution u_ε to the problem (1.2)*

$$u_\varepsilon(x) = PU_{\mu_\varepsilon, \xi_\varepsilon}(x) + \phi_\varepsilon(x), \quad x \in \Omega \setminus B(\xi_0, \varepsilon),$$

where the weight μ_ε of the bubble satisfies

$$\mu_\varepsilon = \varepsilon^{\frac{1}{2}(\frac{N-2}{N-3})} d_\varepsilon \quad \text{for some } d_\varepsilon \rightarrow d \in \mathbb{R}^+,$$

the center ξ_ε of the bubble satisfies

$$\xi_\varepsilon = \xi_0 + \mu_\varepsilon \tau_\varepsilon \quad \text{for some } \tau_\varepsilon \rightarrow \tau \in \mathbb{R}^N$$

and the rest function ϕ_ε is a remainder term.

We would like to point out the difficulties arising in the construction of the solution u_ε . The solution u_ε looks like a bubble concentrating around the center ξ_0 of the removed ball $\overline{B(\xi_0, \varepsilon)}$ as ε goes to zero, so the point where the concentration takes place does not belong to the domain. Its profile resembles a volcano whose crater is the point ξ_0 . Taking into account that the solution concentrates around the hole and at the same time it must satisfy zero Navier boundary condition on the boundary of the hole, it turns to be extremely delicate to study the behavior of the solution in the region around the hole. A key estimate is contained in Proposition 2.1, where the expansion of the projection $PU_{\mu,\xi}$ is performed.

At this stage it is useful to compare the Paneitz-type problem (1.1) with the Yamabe-type problem

$$(1.4) \quad \begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

If the domain Ω is star shaped Pohozaev's identity [20] implies that problem (1.4) has only the trivial solution, while Bahri-Coron [1] proved that (1.4) possesses a solution provided that some homology group of Ω is nontrivial. Again the topology assumption on the domain is not necessary for the existence of solution to (1.4) as proved by Passaseo in [18, 19].

In particular, problem (1.2) is similar to the following one

$$(1.5) \quad \begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} & \text{in } \Omega_\varepsilon := \Omega \setminus \overline{B(\xi_0, \varepsilon)}, \\ u > 0 & \text{in } \Omega_\varepsilon, \\ u = \Delta u = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

which was firstly studied by Coron [6], who proved the existence of a solution provided ε is small enough. If we go deep into the similarities between the two problems (1.2) and (1.5), we could conjecture that all the results obtained for the Yamabe-type problem (1.5) concerning existence of positive and/or sign changing solutions when the domain has one or more small circular holes are also true for the Paneitz-type problem (1.2). In the present paper we only build solutions which concentrate at the center of the hole, whose profile is a single bubble as in Theorem 1.1. We point out that, arguing as Musso-Pistoia [17] (see also Rey [21], Lewandowski [12] and Li-Yan-Yang [13]), it is also possible to get some multiplicity results when the domain has one or more small circular holes. Finally, we believe that arguing exactly as in Musso-Pistoia [16] and Ge-Musso-Pistoia [10] one can construct an arbitrary large number of sign changing solutions whose profile is a superposition of bubbles with alternate sign which concentrate at the center of the hole. Actually, it is worth noting that the expansion of the projection of the bubble given in Proposition 2.1 is the starting point in the construction of all these type of solutions. It is also important to remark that the proof in the case of problem (1.2) could be extremely tedious, even if it can be carried out using the same arguments developed in the study of problem (1.5).

The proof of Theorem 1.1 relies on a very well known Lyapunov-Schmidt reduction. In particular, we will follow the arguments used by Del Pino-Felmer-Musso [8] and Musso-Pistoia in [15]. We shall omit many details on the proof because they can be found, up to some minor modifications, in those papers. We only compute what cannot be deduced from known results. The paper is arranged as follows. Section 2 is devoted to compute the first order approximation of the solution, while Section 3 contains the main steps of the proof of Theorem 1.1.

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2. THE FIRST ORDER APPROXIMATION OF THE SOLUTION

Without loss of generality, we can assume that the center of the hole is the origin, i.e. $0 \in \Omega$ and $\Omega_\varepsilon := \Omega \setminus \overline{B_\varepsilon}$ where $B_\varepsilon := B(0, \varepsilon)$.

We look for a solution to problem (1.2) as

$$(2.1) \quad u_\varepsilon(x) := PU_{\mu, \xi}(x) + \phi_\varepsilon(x),$$

where the weight μ of the bubble and the center ξ of the bubble satisfy

$$(2.2) \quad \mu := d\varepsilon^\sigma \quad \text{and} \quad \xi := \mu\tau, \quad \text{where } d \in \mathbb{R}^+ \cap [\delta, 1/\delta], \quad \tau \in \mathbb{R}^N \cap \overline{B(0, 1/\delta)}$$

for some $\delta > 0$ and the exponent σ is chosen so that (3.19), namely

$$(2.3) \quad \sigma := \frac{N-2}{2(N-3)}.$$

The rest term ϕ_ε is a remainder term which belongs to a suitable space, which will be introduced in the next section.

Our aim is to write the first order approximation of the solution given in (2.1), namely to write the first order approximation of the bubble $PU_{\mu, \xi}$ when μ and ξ satisfy (2.2).

Let G be the Green's function for the bi-Laplacian operator on Ω , that is given $x \in \Omega$

$$(2.4) \quad \begin{cases} \Delta^2 G(x, \cdot) = \gamma_N \delta_x & \text{in } \Omega, \\ G(x, \cdot) = \Delta G(x, \cdot) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\gamma_N := (N-4)(N-2)\text{meas}(\mathbb{S}^{N-1})$. Let H be its regular part, i.e.

$$H(x, y) = \frac{1}{|x-y|^{N-4}} - G(x, y),$$

which verifies

$$\begin{cases} \Delta^2 H(x, \cdot) = 0 & \text{in } \Omega, \\ H(x, \cdot) = \frac{1}{|x-y|^{N-4}} & \text{on } \partial\Omega, \\ \Delta H(x, \cdot) = -2(N-4) \frac{1}{|x-y|^{N-2}} & \text{on } \partial\Omega. \end{cases}$$

The function $v_{\mu,\xi}$ defined by

$$(2.5) \quad v_{\mu,\xi}(y) = U_{\mu,\xi}(y) - PU_{\mu,\xi}(y), \quad y \in \Omega \setminus \overline{B}_\varepsilon,$$

is the unique solution of the problem

$$(2.6) \quad \begin{cases} \Delta^2 v_{\mu,\xi} = 0 & \text{in } \Omega \setminus \overline{B}_\varepsilon, \\ v_{\mu,\xi} = U_{\mu,\xi} & \text{on } \partial(\Omega \setminus \overline{B}_\varepsilon), \\ \Delta v_{\mu,\xi} = \Delta U_{\mu,\xi} & \text{on } \partial(\Omega \setminus \overline{B}_\varepsilon). \end{cases}$$

We introduce the problem

$$(2.7) \quad \begin{cases} \Delta^2 \Upsilon = 0 & \text{in } \mathbb{R}^N \setminus \overline{B}_1, \\ \Upsilon = 2 & \text{on } \partial B_1, \\ \Delta \Upsilon = -2(N-4) & \text{on } \partial B_1, \\ \Upsilon \in \mathcal{D}^{2,2}(\mathbb{R}^N \setminus B_1), \end{cases}$$

which is a sort of limit problem of (2.6) obtained by scaling by ε . It is immediate to check that (2.7) has an unique solution Υ given by

$$(2.8) \quad \Upsilon(x) := \varphi_1 + \varphi_2, \quad \text{where } \varphi_1(x) := \frac{1}{|x|^{N-4}} \quad \text{and} \quad \varphi_2(x) := \frac{1}{|x|^{N-2}}, \quad x \in \mathbb{R}^N \setminus \overline{B}_1.$$

The first order approximation of the function $v_{\mu,\xi}$ defined in (2.5) is given in the following.

Proposition 2.1. *Set*

$$(2.9) \quad R_\varepsilon := PU_{\mu,\xi}(x) - U_{\mu,\xi}(x) + \alpha_N \mu^{\frac{N-4}{2}} H(x, \xi) + a_1 \varphi_1\left(\frac{x}{\varepsilon}\right) + a_2 \varphi_2\left(\frac{x}{\varepsilon}\right),$$

where

$$(2.10) \quad a_1(\varepsilon, d, \tau) := -\frac{\Delta U(\tau)}{2(N-4)} \frac{\varepsilon^2}{\mu^{\frac{N}{2}}}$$

and

$$(2.11) \quad a_2(\varepsilon, d, \tau) := U(\tau) \frac{1}{\mu^{\frac{N-4}{2}}} + \frac{\Delta U(\tau)}{2(N-4)} \frac{\varepsilon^2}{\mu^{\frac{N}{2}}}.$$

Let $\delta > 0$ be fixed and assume that (2.2) holds. There exists a positive constant $C > 0$ such that

$$(2.12) \quad |R_\varepsilon(x)| \leq C \left(\frac{\varepsilon^{N-1}}{\mu^{\frac{N+2}{2}} |x|^{N-4}} + \frac{\varepsilon^{N-1}}{\mu^{\frac{N-2}{2}} |x|^{N-2}} \right) \quad \text{for any } x \in \Omega_\varepsilon.$$

and

$$(2.13) \quad |\Delta R_\varepsilon(x)| \leq C \frac{\varepsilon^{N-1}}{\mu^{\frac{N+2}{2}} |x|^{N-2}} \quad \text{for any } x \in \Omega_\varepsilon.$$

Proof. It is useful to remark that

$$a_1(\varepsilon, d, \tau) = \alpha_N \frac{1}{2} \frac{\varepsilon^2(2|\tau|^2 + N)}{\mu^{\frac{N}{2}}(1 + |\tau|^2)^{\frac{N}{2}}}$$

and

$$a_2(\varepsilon, d, \tau) = \alpha_N \left(\frac{1}{\mu^{\frac{N-4}{2}}(1 + |\tau|^2)^{\frac{N-4}{2}}} - \frac{1}{2} \frac{\varepsilon^2(2|\tau|^2 + N)}{\mu^{\frac{N}{2}}(1 + |\tau|^2)^{\frac{N}{2}}} \right).$$

The function R_ε solves the problem

$$\left\{ \begin{array}{ll} \Delta^2 R_\varepsilon = 0 & \text{in } \Omega \setminus \overline{B_\varepsilon}, \\ R_\varepsilon = \alpha_N \left(-\frac{\mu^{\frac{N-4}{2}}}{(\mu^2 + |x - \xi|^2)^{\frac{N-4}{2}}} + \frac{\mu^{\frac{N-4}{2}}}{|x - \xi|^{N-4}} + a_1 \frac{\varepsilon^{N-4}}{|x|^{N-4}} + a_2 \frac{\varepsilon^{N-2}}{|x|^{N-2}} \right) & \text{on } \partial\Omega, \\ R_\varepsilon = \alpha_N \left(-\frac{\mu^{\frac{N-4}{2}}}{(\mu^2 + |x - \xi|^2)^{\frac{N-4}{2}}} + \mu^{\frac{N-4}{2}} H(x, \xi) + \frac{1}{\mu^{\frac{N-4}{2}} (1 + |\tau|^2)^{\frac{N-4}{2}}} \right) & \text{on } \partial B_\varepsilon, \\ \Delta R_\varepsilon = \alpha_N (N-4) \left(\frac{\mu^{\frac{N-4}{2}} (2|x - \xi|^2 + N\mu^2)}{(\mu^2 + |x - \xi|^2)^{\frac{N}{2}}} - \frac{2\mu^{\frac{N-4}{2}}}{|x - \xi|^{N-2}} - \frac{2|\tau|^2 + N}{\mu^{\frac{N}{2}} (1 + |\tau|^2)^{\frac{N}{2}}} \frac{\varepsilon^{N-2}}{|x|^{N-2}} \right) & \text{on } \partial\Omega, \\ \Delta R_\varepsilon = \alpha_N (N-4) \left(\frac{\mu^{\frac{N-4}{2}} (2|x - \xi|^2 + N\mu^2)}{(\mu^2 + |x - \xi|^2)^{\frac{N}{2}}} + \frac{\mu^{\frac{N-4}{2}}}{N-4} \Delta H(x, \xi) - \frac{2|\tau|^2 + N}{\mu^{\frac{N}{2}} (1 + |\tau|^2)^{\frac{N}{2}}} \right) & \text{on } \partial B_\varepsilon. \end{array} \right.$$

Let us set $\hat{R}_\varepsilon(y) := \mu^{-\frac{N-4}{2}} R_\varepsilon(\varepsilon y)$, $y \in (\varepsilon^{-1}\Omega \setminus \overline{B_1})$. It solves the problem

$$\left\{ \begin{array}{ll} \Delta^2 \hat{R}_\varepsilon = 0 & \text{in } (\varepsilon^{-1}\Omega \setminus \overline{B_1}), \\ \hat{R}_\varepsilon = \alpha_N \left(-\frac{1}{(\mu^2 + |\varepsilon y - \xi|^2)^{\frac{N-4}{2}}} + \frac{1}{|\varepsilon y - \xi|^{N-4}} + \frac{a_1}{\mu^{\frac{N-4}{2}} |y|^{N-4}} + \frac{a_2}{\mu^{\frac{N-4}{2}} |y|^{N-2}} \right) & \text{on } \partial(\varepsilon^{-1}\Omega), \\ \hat{R}_\varepsilon = \alpha_N \left(-\frac{1}{(\mu^2 + |\varepsilon y - \xi|^2)^{\frac{N-4}{2}}} + H(\varepsilon y, \xi) + \frac{1}{\mu^{N-4} (1 + |\tau|^2)^{\frac{N-4}{2}}} \right) & \text{on } \partial B_1, \\ \Delta \hat{R}_\varepsilon = \alpha_N (N-4) \left(\frac{2|\varepsilon y - \xi|^2 + N\mu^2}{(\mu^2 + |\varepsilon y - \xi|^2)^{\frac{N}{2}}} - \frac{2}{|\varepsilon y - \xi|^{N-2}} - \frac{2|\tau|^2 + N}{\mu^{N-2} (1 + |\tau|^2)^{\frac{N}{2}}} \frac{1}{|y|^{N-2}} \right) \varepsilon^2 & \text{on } \partial(\varepsilon^{-1}\Omega), \\ \Delta \hat{R}_\varepsilon = \alpha_N (N-4) \left(\frac{2|\varepsilon y - \xi|^2 + N\mu^2}{(\mu^2 + |\varepsilon y - \xi|^2)^{\frac{N}{2}}} + \frac{1}{N-4} \Delta H(\varepsilon y, \xi) - \frac{2|\tau|^2 + N}{\mu^{N-2} (1 + |\tau|^2)^{\frac{N}{2}}} \right) \varepsilon^2 & \text{on } \partial B_1. \end{array} \right.$$

Moreover, the following estimates hold true for \hat{R}_ε .

$$(2.14) \quad 0 \leq \hat{R}_\varepsilon(y) = O(\mu^2) \quad \forall y \in \partial(\varepsilon^{-1}\Omega),$$

$$(2.15) \quad |\hat{R}_\varepsilon(y)| = O\left(\frac{\varepsilon}{\mu^{N-3}}\right) \quad \forall y \in \partial B_1,$$

$$(2.16) \quad 0 \leq -\Delta \hat{R}_\varepsilon(y) = O(\varepsilon^2 \mu^2) \quad \forall y \in \partial(\varepsilon^{-1}\Omega)$$

and

$$(2.17) \quad |\Delta \hat{R}_\varepsilon(y)| = O\left(\frac{\varepsilon^3}{\mu^{N-1}}\right) \quad \forall y \in \partial B_1.$$

Now, let $R > 0$, $\mathbf{d} := \text{diam } \Omega$ and let Φ be a solution of the problem

$$\left\{ \begin{array}{ll} \Delta^2 \Phi = 0 & \text{in } B_{\varepsilon^{-1}\mathbf{d}} \setminus \overline{B_1}, \\ \Phi = \alpha & \text{on } \partial B_{\varepsilon^{-1}\mathbf{d}}, \\ \Phi = \beta & \text{on } \partial B_1, \\ \Delta \Phi = \alpha' & \text{on } \partial B_{\varepsilon^{-1}\mathbf{d}}, \\ \Delta \Phi = \beta' & \text{on } \partial B_1, \end{array} \right.$$

for some arbitrary numbers α, β, α' and β' . A straightforward computation shows that

$$\Phi(y) = \frac{A}{|y|^{N-4}} + \frac{B}{|y|^{N-2}} + C|y|^2 + D$$

is a solution to such a problem for a suitable choice of A, B, C and D . If we choose

$$\alpha = c\mu^2, \quad \beta = c\frac{\varepsilon^{1-\sigma}}{\mu^{N-4}}, \quad \alpha' = c\varepsilon^2\mu^2, \quad \beta' = c\frac{\varepsilon^{3-\sigma}}{\mu^{N-2}}$$

for some positive constant c , it is easy to check that

$$A = O\left(\frac{\varepsilon^3}{\mu^{N-1}}\right), \quad B = O\left(\frac{\varepsilon}{\mu^{N-3}}\right), \quad C = O(\varepsilon^2\mu^2), \quad D = O(1).$$

Then, by using a comparison argument and taking into account estimates (2.14)-(2.17), we deduce

$$(2.18) \quad |\hat{R}_\varepsilon(y)| \leq M \left(\frac{\varepsilon^3}{\mu^{N-1}|y|^{N-4}} + \frac{\varepsilon}{\mu^{N-3}|y|^{N-2}} + \varepsilon^2\mu^2|y|^2 + \varepsilon \right) \quad \forall y \in \Omega_\varepsilon,$$

and

$$(2.19) \quad |\Delta \hat{R}_\varepsilon(y)| \leq M \left(\frac{\varepsilon^3}{\mu^{N-1}|y|^{N-2}} + \varepsilon^2\mu^2 \right) \quad \forall y \in \Omega_\varepsilon,$$

for some positive constant M , which is independent on ε . Therefore, by (2.18) we deduce (2.12) and by (2.19), we deduce (2.13). \square

3. SCHEME OF THE PROOF

As we said, the proof of Theorem 1.1 relies on a very well known Lyapunov-Schmidt reduction. In this section we will sketch the main steps of the proof skipping many details, because the arguments used in the proof are very similar to the ones used in [8, 15]. The unique new computation is the estimate (3.3) of the reduced energy.

First of all, it is useful to perform a change of variables. Let us consider the expanded domain $\tilde{\Omega}_\varepsilon := \varepsilon^{-\sigma}\Omega_\varepsilon$ and $y = \varepsilon^{-\sigma}x \in \tilde{\Omega}_\varepsilon$, $x \in \Omega_\varepsilon$, where σ is defined in (2.3).

Therefore, u solves the problem (1.1) if and only if the function

$$(3.1) \quad v_\varepsilon(y) = \varepsilon^{\sigma\frac{N-4}{2}} u(\varepsilon^\sigma y), \quad y \in \tilde{\Omega}_\varepsilon$$

solves the problem

$$(3.2) \quad \begin{cases} \Delta^2 v = f(v) & \text{in } \tilde{\Omega}_\varepsilon, \\ v = \Delta v = 0 & \text{on } \partial\tilde{\Omega}_\varepsilon. \end{cases}$$

Here $f(v) = (v^+)^p$ and $p := \frac{N+4}{N-4}$. We point out that the operator Δ^2 with Dirichlet boundary condition satisfies the maximum principle. So any solution to (3.2) is a positive function.

In the expanded variables, the solution we are looking for looks like

$$(3.3) \quad v(y) = V(y) + \tilde{\phi}(y), \quad y \in \tilde{\Omega}_\varepsilon, \quad V(y) := \varepsilon^{\sigma\frac{N-4}{2}} PU_{\mu,\xi}(\varepsilon^\sigma y) \quad \text{and} \quad \tilde{\phi}(y) := \varepsilon^{\sigma\frac{N-4}{2}} \phi(\varepsilon^\sigma y).$$

It is important to point out that the function V is nothing but the projection onto $H^2(\tilde{\Omega}_\varepsilon) \cap H_0^1(\tilde{\Omega}_\varepsilon)$ of the function $\varepsilon^{\sigma\frac{N-4}{2}} U_{\mu,\xi}(\varepsilon^\sigma y) = U_{d,\xi'}(y)$ where we denote by ξ' the point $\varepsilon^{-\sigma}\xi$ and by d the number $\varepsilon^{-\sigma}\mu$ (see (2.2)).

In terms of $\tilde{\phi}$ problem (3.2) rewrites as

$$(3.4) \quad \begin{cases} L(\tilde{\phi}) = N(\tilde{\phi}) + E & \text{in } \tilde{\Omega}_\varepsilon, \\ \tilde{\phi} = \Delta \tilde{\phi} = 0 & \text{on } \partial\tilde{\Omega}_\varepsilon. \end{cases}$$

where the linear operator L is defined by

$$L(\tilde{\phi}) := \Delta^2 \tilde{\phi} - f'(V)\tilde{\phi},$$

the second order term $N(\tilde{\phi})$ is defined by

$$N(\tilde{\phi}) := f(V + \tilde{\phi}) - f(V) - f'(V)\tilde{\phi}$$

and the error term E is defined by

$$(3.5) \quad E := f(V) - f(U_{d,\xi'}).$$

To prove our result we follow the usual strategy of the Lyapunov-Schmidt procedure.

Step 1. We solve a nonlinear problem.

More precisely, given $d > 0$ and $\tau \in \mathbb{R}^N$ (see (2.2)), we find a function $\tilde{\phi} = \tilde{\phi}(\varepsilon, d, \tau)$ such that for some real numbers c_i 's

$$(3.6) \quad \begin{cases} L(\tilde{\phi}) = N(\tilde{\phi}) + E + \sum_{i=0}^N c_i f'(V)W_i & \text{in } \tilde{\Omega}_\varepsilon, \\ \tilde{\phi} = \Delta\tilde{\phi} = 0 & \text{on } \partial\tilde{\Omega}_\varepsilon, \\ \int_{\tilde{\Omega}_\varepsilon} \tilde{\phi}(y) f'(V)(y) W_i(y) dy = 0 & \text{for any } i = 0, 1, \dots, N. \end{cases}$$

The functions W_i are defined as follows. It is known (see [14]) that the set of the solutions to the linearized equation

$$\Delta^2 \vartheta - f'(U_{\mu, \xi}) \vartheta = 0 \quad \text{in } \mathbb{R}^N, \quad \vartheta \in \mathcal{D}^{2,2}(\mathbb{R}^N)$$

is a $(N+1)$ -dimensional linear space spanned by the functions

$$Z_0(x) := \frac{\partial U_{\mu, \xi}}{\partial \mu}(x) = \alpha_N \left(\frac{N-4}{2} \right) \mu^{\frac{N-6}{2}} \frac{|x - \xi|^2 - \mu^2}{(\mu^2 + |x - \xi|^2)^{\frac{N-2}{2}}},$$

$$Z_i(x) := \frac{\partial U_{\mu, \xi}}{\partial \xi_i}(x) = \alpha_N (N-4) \mu^{\frac{N-4}{2}} \frac{x_i - \xi_i}{(\mu^2 + |x - \xi|^2)^{\frac{N-2}{2}}}, \quad i = 1, 2, \dots, N.$$

We denote by PZ_i the projection of Z_i onto $H_0^1(\Omega_\varepsilon) \cap H^2(\Omega_\varepsilon)$ and we set

$$W_i(y) := \varepsilon^{\sigma \frac{N-4}{2}} PZ_i(\varepsilon^\sigma y), \quad y \in \tilde{\Omega}_\varepsilon \quad i = 0, 1, 2, \dots, N.$$

In order to solve problem (3.6) it is necessary to study the linear problem naturally associated to it. More precisely, given $d > 0$ and $\tau \in \mathbb{R}^N$ (see (2.2)) and a function $h \in C^0(\overline{\tilde{\Omega}_\varepsilon})$, find a function $\tilde{\phi}$ such that for some real numbers c_i 's

$$(3.7) \quad \begin{cases} L(\tilde{\phi}) = h + \sum_{i=0}^N c_i f'(V)W_i & \text{in } \tilde{\Omega}_\varepsilon, \\ \tilde{\phi} = \Delta\tilde{\phi} = 0 & \text{on } \partial\tilde{\Omega}_\varepsilon, \\ \int_{\tilde{\Omega}_\varepsilon} \tilde{\phi}(y) f'(V)(y) W_i(y) dy = 0 & \text{for any } i = 0, 1, \dots, N. \end{cases}$$

To study the invertibility of the linear operator L we introduce the L^∞ -weighted spaces $L_*^\infty(\tilde{\Omega}_\varepsilon)$ and $L_{**}^\infty(\tilde{\Omega}_\varepsilon)$ to be, respectively, the spaces of functions defined on $\tilde{\Omega}_\varepsilon$ with finite $\|\cdot\|_*$ and $\|\cdot\|_{**}$ norms defined by

$$\|\eta\|_* = \sup_{y \in \tilde{\Omega}_\varepsilon} \sum_{i=0}^3 \left| (1 + |y - \xi'|^2)^{\frac{2+i}{2}} \sum_{|\alpha|=i} D^\alpha \eta(y) \right|$$

and

$$\|\eta\|_{**} = \sup_{y \in \tilde{\Omega}_\varepsilon} |(1 + |y - \xi'|^2)^4 \eta(y)|.$$

The operator L is uniformly invertible with respect to the above weighted norms provided ε is small enough as it is proved in the next result.

Proposition 3.1. *Let $\delta > 0$ be fixed and assume that (2.2) holds true. Then there exist constants $\varepsilon_0 > 0$ and $C > 0$, such that for every $0 < \varepsilon < \varepsilon_0$ and $h \in C^0(\overline{\tilde{\Omega}_\varepsilon})$, problem (3.7) admits a unique solution $T_\varepsilon(d, \xi', h)$. Furthermore, the map $(d, \xi') \mapsto T_\varepsilon(d, \xi', h)$ is of class C^1 for the $\|\cdot\|_*$ -norm and satisfies*

$$\|T_\varepsilon(d, \xi', h)\|_* \leq C \|h\|_{**}, \quad \|\nabla_{(d, \xi')} T_\varepsilon(d, \xi', h)\|_* \leq C \|h\|_{**}.$$

Moreover,

$$(3.8) \quad |c_i| < C \|h\|_{**} \quad \forall i.$$

Proof. We argue exactly as in Section 5 of [15]. \square

Finally, we have all the ingredients to solve problem (3.2).

Proposition 3.2. *Let $\delta > 0$ be fixed and assume that (2.2) holds true. Then there exist constants $\varepsilon_0 > 0$ and $C > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ there exists a unique solution $\tilde{\phi} = \tilde{\phi}(d, \xi')$ to problem (3.6) such that the map $(d, \xi') \mapsto \tilde{\phi}(d, \xi')$ is of class C^1 for the $\|\cdot\|_*$ -norm and*

$$\|\tilde{\phi}\|_* \leq C\varepsilon^{\frac{(N-2)(N-4)}{2(N-3)}}, \quad \|\nabla_{(d, \xi')} \tilde{\phi}\|_* \leq C\varepsilon^{\frac{(N-2)(N-4)}{2(N-3)}}.$$

Proof. We argue exactly as in Section 6 of [15], once one has the estimate of the error term E defined in (3.5). Indeed by Proposition 2.1 we deduce $\|E\|_{**} = O(\varepsilon^{\frac{(N-2)(N-4)}{2(N-3)}})$. \square

Step 2. We reduce the problem to a finite dimensional one.

Let us consider the function $J_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^N \rightarrow \mathbb{R}$ defined by

$$(3.9) \quad J_\varepsilon(d, \tau) := I_\varepsilon(V + \tilde{\phi}),$$

where ϕ is the function found in Proposition 3.2 and the functional $I_\varepsilon : H^2(\tilde{\Omega}_\varepsilon) \cap H_0^1(\tilde{\Omega}_\varepsilon) \rightarrow \mathbb{R}$ is defined by

$$I_\varepsilon(v) := \frac{1}{2} \int_{\tilde{\Omega}_\varepsilon} |\Delta v|^2 dy - \frac{1}{p+1} \int_{\tilde{\Omega}_\varepsilon} (v^+)^{p+1} dy.$$

Proposition 3.3. *(i) The function $v = V + \tilde{\phi}$ is a solution to problem (3.4), namely $c_i = 0$ in (3.6) for all i 's, if and only if (d, ξ) is a critical point of J_ε .
(ii) It holds true that*

$$J_\varepsilon(d, \tau) = a_N + \varepsilon^{\frac{(N-2)(N-4)}{2(N-3)}} \Psi(d, \tau) + o\left(\varepsilon^{\frac{(N-2)(N-4)}{2(N-3)}}\right),$$

C^1 -uniformly with respect to (d, τ) in compact sets of $\mathbb{R}^+ \times \mathbb{R}^n$. Here

$$(3.10) \quad \Psi(d, \tau) := -b_N \Delta U(\tau) U(\tau) \frac{1}{d^{N-2}} + c_N H(0, 0) d^{N-4},$$

where $U = U_{0,1}$ and a_N, b_N and c_N are positive constants defined by

$$a_N := \int_{\mathbb{R}^N} U^{\frac{2N}{N-4}}(y) dy, \quad b_N := \frac{3}{4}(N-2) \text{meas}(\mathbb{S}^{N-1}) \quad c_N := \frac{1}{2} \alpha_N \int_{\mathbb{R}^N} U^{\frac{N+4}{N-4}}(y) dy.$$

Proof. We argue exactly as in Section 4 of [15]. We only need to compute the leading term in the expansion of $J_\varepsilon(d, \xi)$, which is nothing but the energy of the bubble, namely $I_\varepsilon(V)$. So we have to compute:

$$(3.11) \quad \begin{aligned} I_\varepsilon(V) &= \frac{1}{2} \int_{\tilde{\Omega}_\varepsilon} |\Delta V|^2 - \frac{1}{p+1} \int_{\tilde{\Omega}_\varepsilon} V^{p+1} \\ &= \frac{1}{2} \int_{\Omega_\varepsilon} |\Delta P U_{\mu, \xi}|^2 - \frac{1}{p+1} \int_{\Omega_\varepsilon} (P U_{\mu, \xi})^{p+1} \\ &= \frac{1}{2} \int_{\Omega_\varepsilon} U_{\mu, \xi}^p P U_{\mu, \xi} - \frac{1}{p+1} \int_{\Omega_\varepsilon} (P U_{\mu, \xi})^{p+1} \\ &= \frac{2}{N} \int_{\Omega_\varepsilon} U_{\mu, \xi}^{p+1} - \frac{1}{2} \int_{\Omega_\varepsilon} U_{\mu, \xi}^p (P U_{\mu, \xi} - U_{\mu, \xi}) \\ &\quad - \frac{1}{2} p \int_{\Omega_\varepsilon} (t P U_{\mu, \xi} + (1-t) U_{\mu, \xi})^{p-1} (P U_{\mu, \xi} - U_{\mu, \xi})^2. \end{aligned}$$

The first term in the R.H.S. of (3.11) is estimated as follows.

$$(3.12) \quad \begin{aligned} \int_{\Omega \setminus \bar{B}_\varepsilon} U_{\mu, \xi}^{p+1} &= \alpha_N^{\frac{2N}{N-4}} \int_{\Omega \setminus \bar{B}_\varepsilon} \frac{\mu^N}{(\mu^2 + |x - \xi|^2)^N} dx \\ &= \alpha_N^{\frac{2N}{N-4}} \int_{\mu^{-1}(\Omega \setminus \bar{B}_\varepsilon)} \frac{1}{(1 + |y - \tau|^2)^N} dy \\ &= \alpha_N^{\frac{2N}{N-4}} \int_{\mathbb{R}^N} \frac{1}{(1 + |y|^2)^N} dy + O\left(\left(\frac{\varepsilon}{\mu}\right)^N + \mu^N\right). \end{aligned}$$

The second term in the R.H.S. of (3.11) is estimated as follows. By Proposition 2.1 we get

$$(3.13) \quad \begin{aligned} & \int_{\Omega \setminus \overline{B_\varepsilon}} U_{\mu, \xi}^p (PU_{\mu, \xi} - U_{\mu, \xi}) = \int_{\Omega \setminus \overline{B_\varepsilon}} U_{\mu, \xi}^p R_\varepsilon \\ & - \int_{\Omega \setminus \overline{B_\varepsilon}} U_{\mu, \xi}^p \left(\alpha_N \mu^{\frac{N-4}{2}} H(x, \xi) + a_1 \varphi_1 \left(\frac{x}{\varepsilon} \right) + a_2 \varphi_2 \left(\frac{x}{\varepsilon} \right) \right) dx, \end{aligned}$$

where φ_1 and φ_2 are the functions defined in (2.8), and a_1 and a_2 are the constants defined in (2.10) and (2.11). We estimate each summand in the right hand side of (3.13). We scale $x - \xi = \mu y$ and we get

$$(3.14) \quad \begin{aligned} & \int_{\Omega \setminus \overline{B_\varepsilon}} U_{\mu, \xi}^p \alpha_N \mu^{\frac{N-4}{2}} H(x, \xi) dx \\ & = \alpha_N^{\frac{2N}{N-4}} \int_{\mu^{-1}(\Omega \setminus (\overline{B_\varepsilon} - \xi))} \mu^{N-4} H(\mu y + \xi, \xi) \frac{1}{(1 + |y|^2)^{\frac{N+4}{2}}} dy \\ & = \alpha_N^{\frac{2N}{N-4}} \mu^{N-4} H(0, 0) \left(\int_{\mathbb{R}^N} \frac{1}{(1 + |y|^2)^{\frac{N+4}{2}}} dy + o(1) \right), \end{aligned}$$

$$(3.15) \quad \begin{aligned} & \int_{\Omega \setminus \overline{B_\varepsilon}} U_{\mu, \xi}^p a_1 \varphi_1 \left(\frac{x}{\varepsilon} \right) dx \\ & = - \frac{\Delta U(\tau)}{2(N-4)} \frac{\varepsilon^2}{\mu^{\frac{N}{2}}} \mu^{\frac{N-4}{2}} \int_{\mu^{-1}(\Omega \setminus (\overline{B_\varepsilon} - \xi))} \varphi_1 \left(\frac{\mu}{\varepsilon} (y + \tau) \right) U^p(y) dy \\ & = \left(\frac{\varepsilon}{\mu} \right)^{N-2} \left(- \frac{\Delta U(\tau)}{2(N-4)} \int_{\mathbb{R}^N} \frac{1}{|y + \tau|^{N-4}} U^p(y) dy + o(1) \right) \\ & = \left(\frac{\varepsilon}{\mu} \right)^{N-2} \left(- \frac{N-2}{2} \text{meas}(\mathbb{S}^{N-1}) \Delta U(\tau) U(\tau) + o(1) \right), \end{aligned}$$

because the function U solves (1.3) and the Green's function of Δ^2 in $\mathcal{D}^{2,2}(\mathbb{R}^N)$ is $\frac{1}{|x-y|^{N-4}}$ with the normalization constant given by $(N-2)(N-4)\text{meas}(\mathbb{S}^{N-1})$ (see also (2.4)),

$$(3.16) \quad \begin{aligned} & \int_{\Omega \setminus \overline{B_\varepsilon}} U_{\mu, \xi}^p a_2 \varphi_2 \left(\frac{x}{\varepsilon} \right) dx \\ & = U(\tau) \int_{\mu^{-1}(\Omega \setminus (\overline{B_\varepsilon} - \xi))} \varphi_2 \left(\frac{\mu}{\varepsilon} (y + \tau) \right) U^p(y) dy \\ & + \frac{\Delta U(\tau)}{2(N-4)} \frac{\varepsilon^2}{\mu^{\frac{N}{2}}} \mu^{\frac{N-4}{2}} \int_{\mu^{-1}(\Omega \setminus (\overline{B_\varepsilon} - \xi))} \varphi_2 \left(\frac{\mu}{\varepsilon} (y + \tau) \right) U^p(y) dy \\ & = \left(\frac{\varepsilon}{\mu} \right)^{N-2} \left(U(\tau) \int_{\mathbb{R}^N} \frac{1}{|y + \tau|^{N-2}} U^p(y) dy + o(1) \right) \\ & + \left(\frac{\varepsilon}{\mu} \right)^N \left(\frac{\Delta U(\tau)}{2(N-4)} \int_{\mathbb{R}^N} \frac{1}{|y + \tau|^{N-2}} U^p(y) dy + o(1) \right), \\ & = \left(\frac{\varepsilon}{\mu} \right)^{N-2} \left(U(\tau) \int_{\mathbb{R}^N} \frac{1}{|y + \tau|^{N-2}} U^p(y) dy + o(1) \right) \\ & = \left(\frac{\varepsilon}{\mu} \right)^{N-2} \left(-(N-2)\text{meas}(\mathbb{S}^{N-1}) U(\tau) \Delta U(\tau) + o(1) \right), \end{aligned}$$

because the function $W = \Delta U$ solves the problem $\Delta W = U^p$ in \mathbb{R}^N and the Green's function of $-\Delta$ in $\mathcal{D}^{2,2}(\mathbb{R}^N)$ is $\frac{1}{|x-y|^{N-2}}$ with the normalization constant given by $(N-2)\text{meas}(\mathbb{S}^{N-1})$.

Moreover, by (2.12) we deduce

$$\begin{aligned}
 & \int_{\Omega \setminus \bar{B}_\varepsilon} U_{\mu,\xi}^p |R_\varepsilon(x)| dx \\
 (3.17) \quad &= O \left(\int_{\Omega \setminus \bar{B}_\varepsilon} \frac{\mu^{\frac{N+4}{2}}}{(\mu^2 + |x - \xi|^2)^{\frac{N+4}{2}}} \left(\frac{\varepsilon^{N-1}}{\mu^{\frac{N+2}{2}}} \frac{1}{|x|^{N-4}} + \frac{\varepsilon^{N-1}}{\mu^{\frac{N-2}{2}}} \frac{1}{|x|^{N-2}} \right) dx \right) \\
 &= O \left(\frac{\varepsilon^{N-1}}{\mu^{N-2}} \int_{\mathbb{R}^N} \frac{1}{(1 + |y|^2)^{\frac{N+4}{2}}} \frac{1}{|y|^{N-4}} dy + \frac{\varepsilon^{N-1}}{\mu^{N-1}} \int_{\mathbb{R}^N} \frac{1}{(1 + |y|^2)^{\frac{N+4}{2}}} \frac{1}{|y|^{N-2}} dy \right) \\
 &= o \left(\frac{\varepsilon^{N-2}}{\mu^{N-2}} \right)
 \end{aligned}$$

The last term in the R.H.S. of (3.11) can be estimated as (3.17) and so

$$(3.18) \quad \frac{1}{2} p \int_{\Omega_\varepsilon} (tPU_{\mu,\xi} + (1-t)U_{\mu,\xi})^{p-1} (PU_{\mu,\xi} - U_{\mu,\xi})^2 = o \left(\frac{\varepsilon^{N-2}}{\mu^{N-2}} \right).$$

We collect all the estimates (3.11)–(3.18) and the claim follows, provided σ is chosen so that

$$(3.19) \quad \mu^{N-4} \sim \left(\frac{\varepsilon}{\mu} \right)^{N-2} \Rightarrow \mu^{2(N-3)} \sim \varepsilon^{N-2} \Rightarrow \sigma = \frac{N-2}{2(N-3)}.$$

□

Proof of the Theorem 1.1. We know that $PU_{\mu,\xi} + \phi$ is a solution to problem (1.2) if and only if the function $V + \tilde{\phi}$ is a solution of (3.2). From (i) of Proposition 3.3 we have that the function $V + \tilde{\phi}$ is a solution of (3.2) or (3.4) if and only if (d, τ) is a critical point of the reduced energy J_ε defined in (3.9). Then, from (ii) of Proposition 3.3, we only need to find a critical point of the function Ψ defined in (3.10), which is stable under C^1 perturbations. Indeed, it is easy to check that the function Ψ has a nondegenerate critical point $\left(-\frac{(N-2)b_N \Delta U(0)U(0)}{(N-4)c_N H(0,0)}, 0 \right)$ of “saddle” type, which is stable with respect to C^1 perturbations. That proves our claim. □

REFERENCES

- [1] A. Bahri and J.M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain*, Comm. Pure Appl. Math. **41** (1988), 255–294.
- [2] T. Branson, S.-Y.A. Chang, P. Yang, *Estimates and extremal problems for the log-determinant on 4-manifolds*, Commun. Math. Phys. **149** (1992) 241–262.
- [3] S.-Y.A. Chang, *On Paneitz operator a fourth order differential operator in conformal geometry*, Harmonic Analysis and Partial Differential Equations; essays in honour of A.P. Calderon; Chicago Lectures in Mathematics, 1999, Chap. 8, pp. 127–150.
- [4] S.-Y.A. Chang, M. Gursky, P. Yang, *On the regularity of a fourth order PDE with critical exponent*, Am. J. Math. **121** (1999) 215–257.
- [5] S.-Y.A. Chang, P. Yang, *Extremal metrics of zeta functional determinants on 4-manifolds*, Ann. Math. **142** (1995) 171–212.
- [6] J.M. Coron, *Topologie et cas limite des injections de Sobolev*, C. R. Acad. Sci. Paris Sr. I Math. **299** (1984), 209–212.
- [7] F. Ebobisse, M. Ahmedou, *On a nonlinear fourth order elliptic equation involving the critical Sobolev exponent*, Nonlinear Anal. **52** (2003), 1535–1552.
- [8] M. Del Pino, P. Felmer, M. Musso, *Two-bubble solutions in the super-critical Bahri-Coron’s problem*, Calc. Var. Partial Differential Equations **16** (2003), 113–145.
- [9] F. Gazzola, H.-C. Grunau, M. Squassina, *Existence and nonexistence results for critical growth biharmonic elliptic equations*, Calc. Var. Partial Differential Equations **18** (2003), 117–143.
- [10] Y. Ge, M. Musso, A. Pistoia, *Sign changing tower of bubbles for an elliptic problem at the critical exponent in pierced non-symmetric domains* Comm. Partial Differential Equations **35** (2010), no. 8, 1419–1457.
- [11] C. S. Lin, *A classification of solutions of a conformally invariant fourth order equation in \mathbb{R}^n* , Comment. Math. Helv. **73** (1998), 206–231.
- [12] R. Lewandowski, *Little holes and convergence of solutions of $-\Delta u = u^{(N+2)/(N-2)}$* , Nonlinear Anal. **14** (1990), 873–888.

- [13] G. Li, S. Yan, J. Yang, *An elliptic problem with critical growth in domains with shrinking holes*, J. Differential Equations **198** (2) (2004) 275300.
- [14] G. Lu, J. Wei, *On a Sobolev inequality with remainder terms*, Proc. Amer. Math. Soc. **128**(2000), 75–84.
- [15] M. Musso, A. Pistoia, *Persistence of Coron's solution in nearly critical problems* Ann. Scuola Norm. Sup. Pisa Cl. Sci **5** (2007), 331–357.
- [16] M. Musso, A. Pistoia, *Sign changing solutions to a nonlinear elliptic problem involving the critical Sobolev exponent in pierced domains* J. Math. Pures Appl. **86** (2006), no. 6, 510–528.
- [17] M. Musso, A. Pistoia, *Sign changing solutions to a Bahri-Coron's problem in pierced domains*. Discrete Contin. Dyn. Syst. **21** (2008), no. 1, 295–306.
- [18] D. Passaseo, *New nonexistence results for elliptic equations with supercritical nonlinearity*, Differential Integral Equations **8** (1995), 577–586.
- [19] D. Passaseo, *Nontrivial solutions of elliptic equations with supercritical exponent in contractible domains*, Duke Math. J. **92** (1998), 429–457.
- [20] S. I. Pohožaev, *On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$* , (Russian) Dokl. Akad. Nauk SSSR **165** (1965), 36–39.
- [21] O. Rey, *Sur un problème variationnel non compact: l'effet de petits trous dans le domaine*, C. R. Acad. Sci. Paris **308** (1989) 349–352.
- [22] R.C.A.M. Van der Vorst, *Best constant for the embedding of the space $H^2 \cap H_0^1(\Omega)$ into $L^{2N/(N-4)}(\Omega)$* , Differential Integral Equations **6** (1993), 259–276.

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