KELLER-OSSERMAN TYPE CONDITIONS FOR SOME ELLIPTIC PROBLEMS WITH GRADIENT TERMS

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Abstract. In this paper we consider the elliptic boundary blow-up problems

$$\begin{cases} \Delta u \pm g(|\nabla u|) = f(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain and the functions f and g are increasing and continuous. Our main concern will be to prove both existence and nonexistence of nonnegative solutions, depending on new integral conditions of Keller-Osserman type involving f and g. We show in particular that the problem with a minus sign may have solutions inclusive for some functions g with slightly superquadratic growth at infinity that is somehow not expected. We also obtain uniqueness of nonnegative solutions in some cases.

1. Introduction

The main objective of this paper is to analyze the existence and nonexistence of nonnegative solutions to the problems

$$(P_\pm) \qquad \begin{cases} \Delta u \pm g(|\nabla u|) = f(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases}$$

where Ω is a C^2 bounded domain of $\mathbb{R}^N$ and the functions f, g are continuous and increasing, with $f(0) = g(0) = 0$. By a solution to $(P_\pm)$ we mean a function $u \in C^1_{\text{loc}}(\Omega)$ which verifies the equation in the weak sense and $u(x) \to \infty$ as $d(x) := \text{dist}(x, \partial\Omega) \to 0$.

The model problem without the gradient terms, namely

$$(1.1) \qquad \begin{cases} \Delta u = f(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases}$$

has generated a good deal of research. The important case $f(u) = e^u$ was analyzed in the early works of Bieberbach [9] when $N = 2$ and Rademacher [49] for $N = 3$, and later reconsidered in [36]. The other significant example $f(u) = u^p$, $p > 1$, was dealt with in [47], [41] and [32] (also in [17] when the Laplacian is replaced by the p-Laplacian Δ_p). These cases have been also studied when the underlying domain Ω is not necessarily smooth: see [44], [45] and [18].

As for general increasing nonlinearities $f(u)$, it has been known since the pioneering works of Keller [31] and Osserman [46] that problem (1.1) admits a solution if and only if the nowadays called Keller-Osserman condition holds, i. e.

$$(1.2) \qquad \int_1^\infty \frac{ds}{\sqrt{F(s)}} < \infty,$$

where $F(s) = \int_0^s f(t)dt$ (see also [20] when f is not increasing). Later, other important questions concerning boundary behavior of solutions and uniqueness or multiplicity, have been considered in many works. We quote for instance [5], [37] and [25] for uniqueness for general nonlinearities and [1], [2] for multiplicity results. See also [7], [8], [37], [15] and [3], where more precise information on the asymptotic behavior of the solutions is obtained. A great amount of works have been also interested in the appearance of weights in the equation, both vanishing on $\partial\Omega$ ([19], [26], [13], [14], [12], [42], [43], [24]) or singular at $\partial\Omega$ ([10], [11], [23]). More references can be found in the survey [50].

With regard to boundary blow-up problems containing gradient terms, Lasry and Lions [35] considered the following:

$$\begin{cases} \Delta u - |\nabla u|^p = \lambda u + h & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $p > 1$, $\lambda > 0$ and h is smooth in Ω (it appears in stochastic control problems with state constraints). Assuming h has a prescribed growth near $\partial\Omega$, it was shown in [35] that there exists a unique solution to (1.3) when $1 < p \leq 2$. Further developments of this and related problems have been made in [48], [38], [39]. In particular, in this paper we can extend the results in [35] when we consider $h = 0$ and we replace the term $|\nabla u|^p$ by $g(|\nabla u|) = |\nabla u|^2 \log(|\nabla u| + 1)$ which is slightly superquadratic. We believe that this can be extended to cover the case of nontrivial h.

Another type of problems including gradient terms in the equation was proposed by Bandle and Giarrusso in [4]. They considered

$$\begin{cases} \Delta u \pm |\nabla u|^q = f(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

for $q > 0$ and general differentiable functions f, paying special attention to the two "classical" nonlinearities $f(u) = u^p$, $p > 1$ and $f(u) = e^u$. Some existence results were obtained, together with boundary behavior of positive solutions in most cases. This problem was later analyzed again in [28] and [29] (see also some extensions to problems containing weights in [27], [34], [52], [53] and [54]).

However, at the best of our knowledge, no results are available for the general problems $(P_\pm)$ aside the case $g(t) = t^q$, $q > 0$. Thus in the present paper, and under quite general assumptions on f and g, we will obtain conditions to ensure existence or nonexistence of nonnegative solutions. Let us mention that our conditions are not sharp, due to the fact that the one dimensional version of $(P_\pm)$ is not integrable, but nevertheless they are in the case of power nonlinearities $f(t) = t^p$, $g(t) = t^q$, $p, q > 0$.

We would like to stress that for problem (1.4) with the minus sign, the results in [4] show that the exponent $q = 2$ is critical in some sense, since no solutions are expected to exist when $q > 2$ (although this fact is not proved there for smooth bounded domains). Among other things, we show that for problem (P_-) it is possible to have solutions for nonlinearities which grow faster than quadratically, the essential feature being an integrability condition on $s/g(s)$ at infinity.

In our results about existence we need to obtain interior bounds for the gradient of solutions. In order to find these bounds we will assume one of the following two conditions on g:

(a) There exists $t_0 > 0$ such that g is differentiable for $t \geq t_0$ and

$$\frac{g'(t)}{g(t)^2} \leq C\, t^{-\gamma}, \quad t \geq t_0$$

for some $\gamma > 2$, or

(b) There exists a positive constant C such that $|g(t)| \leq Ct^2$, for large t.

Note that condition (a) is verified by the case $g(s) = s^2\log(s+1)$ while (b) is the standard condition to get this kind of interior bounds.

Before proceeding to state our principal results below, it is important to stress that in all of them we will assume the following general hypotheses on f and g:

$$(f_0 - g_0) \qquad \begin{array}{c} f \text{ and } g \text{ are continuous increasing functions} \\ \text{such that } f(0) = g(0) = 0. \end{array}$$

A word of caution: we are always concerned with nonnegative solutions, but since f and g are only continuous, the strict positivity of the solutions cannot be guaranteed, that is, the strong maximum principle is not always valid (cf. [22] and [21] for its validity in this context). In our present situation the solutions could have a "dead core" in the interior of the domain, but this is not essential when dealing with existence and uniqueness.

We start considering problem (P_+), and introduce

$$\Gamma(s) = \int_0^{2s} g(t)\,dt + 2Ns^2. \tag{1.5}$$

Then we have the following existence/nonexistence result.

Theorem 1. *Let f and g be functions satisfying $(f_0 - g_0)$. Then:*

(i) *If f does not verify the Keller-Osserman condition* (1.2) *or if*

$$\int_1^{\infty} \frac{ds}{g^{-1}\big(f(s)\big)} = \infty, \tag{1.6}$$

then problem (P_+) does not admit nonnegative solutions.

(ii) *If g verifies one of the conditions (a) or (b) above and*

$$\int_1^{\infty} \frac{ds}{\Gamma^{-1}\big(\frac{1}{2}F(s)\big)} < \infty, \tag{1.7}$$

where Γ is given by (1.5)*, then there exists at least a nonnegative weak solution u to (P_+), which in addition verifies $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$ for every $\alpha \in (0,1)$.*

Let us now turn to problem (P_-). We have already mentioned that the power $q = 2$ is critical among power-like nonlinearities regarding existence. In the general setting, we will prove that the "criticality" of g for existence of solutions with large boundary condition depends on the divergence of the integral $\int_1^\infty \frac{s}{g(s)}ds$, that is

$$\int_1^{\infty} \frac{s}{g(s)} ds = \infty. \tag{1.8}$$

Theorem 2. *Let f and g be functions satisfying $(f_0 - g_0)$, and assume that g verifies condition* (1.8)*. Then:*

(i) *If*

$$\int_1^{\infty} \frac{ds}{f(s) + g(s)} = \infty, \tag{1.9}$$

then problem (P_-) does not have nonnegative solutions.

(ii) *If f verifies the Keller-Osserman condition* (1.2) *or f verifies $\lim_{t\to\infty} f(t) = \infty$ and g verifies one of the conditions (a) or (b) above and is such that*

$$\int_1^{\infty} \frac{ds}{g(s)} < \infty. \tag{1.10}$$

then problem (P_-) *admits at least a nonnegative weak solution* u*, which verifies* $u \in C^{1,\alpha}_{\rm loc}(\Omega)$ *for every* $\alpha \in (0,1)$.

Remark 1. Some relations between the different conditions appeared above are expected to hold. For instance, it is easy to check that (1.9) implies (1.8), while (1.8) is also implied by (b). On the other hand, (a) with the additional condition $\lim_{t\to\infty} g(t) = \infty$ gives (1.10).

Our proofs of existence and nonexistence for nonnegative solutions to $(P_\pm)$ rely in comparison with solutions to the same problems in balls of $\mathbb{R}^N$. Thus it is important to analyze the radial version of those problems, which is in turn close to the one-dimensional version. Since these equations are not integrable, our method inspired in [21] will be to compare them with some integrable ones and this leads to the previous integral conditions of Keller-Osserman type involving f and g.

In this work we also are able to prove that condition (1.8) is sharp in the sense that non existence holds not only for (P$_-$) but also for the problem

$$\begin{cases} \Delta u - g(|\nabla u|) = f(u) & \text{in } \Omega, \\ u = n & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

when n is large enough, provided that the integral in (1.8) converges, see Theorem 6 in Section 5.

Also we find some *uniqueness* results for nonnegative solutions to (P_+) and (P_-) under some extra assumptions on f and g, which are used to obtain the precise asymptotic behavior of all solutions near the boundary, and are essentially the usual hypotheses in the literature when $g = 0$ (cf. for instance [7]). See Theorems 8 and 9, respectively, in Section 6. All our results are illustrated by means of examples. Moreover, we complete the study in [4] of the particular choice $f(s) = s^p$ and $g(s) = s^q$, proving uniqueness when one has existence and finding a new range for the exponent p and q where non existence holds, see Corollary 13 in Section 7.

The rest of the paper is organized as follows: in Section 2 we deal with some preliminary properties of radial solutions to the Cauchy problems related to $(P_\pm)$. Section 3 is devoted to obtain existence of solutions to the finite boundary value problems associated to $(P_\pm)$ while in Section 4 the existence and nonexistence results are considered. In Section 5 we show that condition (1.8) is necessary for existence in problems (P_-) and (1.11). The uniqueness issue is undertaken in Section 6 while in Section 7 some illustrative examples are analyzed. Finally we include an Appendix where we prove some interior gradient bounds for solutions to $(P_\pm)$.

2. Properties of radial solutions

In this section we are going to prove some preliminary properties of solutions to the Cauchy problem

$$\begin{cases} u'' + \dfrac{N-1}{r} u' = f(u) \pm g(u'), \\ u(0) = u_0, \ u'(0) = 0, \end{cases} \tag{2.1}$$

where f and g are functions satisfying $(f_0 - g_0)$, and $u_0 \geq 0$. By continuity, it is well known that there exists at least a solution to (2.1). We are interested only in nonnegative solutions.

Our main result in this section is the following:

Proposition 3. *Let u be a nonnegative solution to (2.1). If $u_0 > 0$ then $u'(r) > 0$, $u''(r) \geq 0$ for $r > 0$. If $u_0 = 0$, then there exists $r_0 \geq 0$ such that $u \equiv 0$ in $[0, r_0]$, $u'(r) > 0$, $u''(r) \geq 0$ for $r > r_0$. In particular, u and u' are increasing functions and for every $R > 0$ such that u is defined in $(0, R)$,*

$$u(r) \leq u_0 + Ru'(r), \quad r \in (0, R). \tag{2.2}$$

Remark 2. *Note that if f and g are locally Lipschitz functions, then $r_0 = 0$.*

Proof. Assume first $u_0 > 0$. From the equation we obtain $u''(0) = \frac{1}{N} f(u_0) > 0$, so that $u''(r) > 0$ if $r > 0$, r close enough to zero. This implies $u'(r) > 0$ for $r > 0$ close enough to zero. Assume there exists $r_1 > 0$ with $u'(r) > 0$ if $r \in (0, r_1)$ and $u'(r_1) = 0$. Then we would have $u''(r_1) \leq 0$ and the equation would give

$$u''(r_1) = f\big(u(r_1)\big) > 0,$$

a contradiction. Hence $u'(r) > 0$ for $r > 0$.

Suppose next $u_0 = 0$. Define

$$r_0 = \sup\{\tilde{r} : \ u(r) = 0 \text{ in } [0, \tilde{r}]\},$$

and let us prove that $u'(r) > 0$ when $r > r_0$. Notice first that there exists a sequence $r_n \downarrow r_0$ such that $u'(r_n) > 0$. If not, we would have $u' \leq 0$ in $[r_0, r_0 + \varepsilon]$ for some $\varepsilon > 0$ and this leads to $u = 0$ in $[r_0, r_0 + \varepsilon]$, contradicting the definition of r_0.

We may assume $u(r_n) > 0$, since in the case $u(r_n) = 0$ we could take $\bar{r}_n > r_n$, close to r_n with $u(\bar{r}_n) > 0$. Now a similar reasoning as before shows that $u'(r) > 0$ if $r > r_n$. Hence $u'(r) > 0$ for $r > r_0$.

Let us now deal with the sign of u''. We need different proofs for problem (2.1) with a plus or with a minus sign. Let us begin with the minus sign:

$$u'' + \frac{N-1}{r} u' = f(u) - g(u').$$

Assume $u''(r_2) < 0$ for some $r_2 > 0$. Let

$$\tilde{r}_0 = \inf\{\tilde{r} : \ u''(r) < 0 \text{ in } (\tilde{r}, r_2)\}.$$

Since $u''(0) \geq 0$, we have $u''(\tilde{r}_0) = 0$. Moreover, $u''(r) < 0$ for $r > \tilde{r}_0$, r close to $\tilde{r}_0$. This implies that u' is decreasing for $r \geq \tilde{r}_0$, r close to $\tilde{r}_0$. Since u is increasing, we have that

$$u'' = f(u) - \frac{N-1}{r} u' - g(u')$$

is increasing. But then $u''(\tilde{r}_0) = 0$ implies that $u'' > 0$ if $r > \tilde{r}_0$, r close to $\tilde{r}_0$, a contradiction. Thus $u'' \geq 0$.

Next, let us deal with the plus sign:

$$u'' + \frac{N-1}{r} u' = f(u) + g(u').$$

There exists $r_3 > 0$ such that $u''(r_3) > 0$. If not, we would always have $u'' \leq 0$ and since $u'(0) = 0$, we arrive at $u' \leq 0$, which is impossible. Let us see that $u''(r) > 0$ if $r > r_3$. Indeed, if there exists $r_4 > r_3$ such that $u''(r_4) = 0$, we have for small $h > 0$:

$$\frac{N-1}{r_4} u'(r_4) = f\big(u(r_4)\big) + g\big(u'(r_4)\big)$$

and

$$u''(r_4 - h) + \frac{N-1}{r_4 - h} u'(r_4 - h) = f\big(u(r_4 - h)\big) + g\big(u'(r_4 - h)\big),$$

so that subtracting and dividing by $-h$:

$$\frac{u''(r_4-h)-u''(r_4)}{-h}+\frac{N-1}{-h}\left(\frac{u'(r_4-h)}{r_4-h}-\frac{u'(r_4)}{r_4}\right)$$
$$=\frac{f\big(u(r_4-h)\big)-f\big(u(r_4)\big)}{-h}+\frac{g\big(u'(r_4-h)\big)-g\big(u'(r_4)\big)}{-h}.$$

Taking into account that u is increasing and u' decreasing for $r\in(r_3,r_4)$, we arrive at

$$\frac{1}{-h}\left(\frac{u'(r_4-h)}{r_4-h}-\frac{u'(r_4)}{r_4}\right)\geq 0,$$

and letting $h\to 0$, this leads to

$$-\frac{u'(r_4)}{r_4^2}\geq 0,$$

which is impossible. Hence $u''>0$ if $r>r_0$ in this case as well.

Now take r_0 so that $u=0$ in $[0,r_0]$ and $u>0$ for $r>r_0$ (of course $r_0=0$ is possible). There exists $r_n\downarrow r_0$ such that $u''(r_n)>0$, since on the contrary we would have $u''\leq 0$ in $[r_0,r_0+\varepsilon]$ for some small ε and this implies $u'\leq 0$ and $u=0$ in $[r_0,r_0+\varepsilon]$, which is not possible. By the previous proof, $u''(r)>0$ if $r>r_n$, and hence $u''(r)>0$ for $r>r_0$. Thus $u''(r)\geq 0$ if $r\geq 0$.

Finally, since $u''\geq 0$, we have that u' is nondecreasing. Thus, for $r\in(0,R)$:

$$u(r)=u_0+\int_0^r u'(s)\,ds\leq u_0+Ru'(r).$$

This concludes the proof. □

3. Existence of solutions with finite datum

In this section we assume that f and g are functions satisfying (f_0-g_0). Here we will prove that, under suitable conditions on the growth of g, there always exists a solution to the problems with finite datum

$$(P^n_\pm)\qquad \begin{cases}\Delta u\pm g(|\nabla u|)=f(u) & \text{in } \Omega,\\ u=n & \text{on } \partial\Omega,\end{cases}$$

where $n\in\mathbb{N}$. We warn the reader that g may have a superquadratic growth in the gradient, thus to ensure existence of solutions we will truncate the nonlinearity, and then look for right bounds for the gradient, in the spirit of [40] (see also [4]).

Since in what follows it will be easier to work with $|\nabla u|^2$ instead of $|\nabla u|$, we denote $\tilde{g}(t)=g(\sqrt{t})$ for $t\geq 0$. Now choose $M>0$ and let g_M be an increasing, bounded function such that $g_M(t)=\tilde{g}(t)$ for $t\in[0,M]$. Consider the truncated problems

$$(3.1)\qquad \begin{cases}\Delta u\pm g_M(|\nabla u|^2)=f(u) & \text{in } \Omega,\\ u=n & \text{on } \partial\Omega.\end{cases}$$

Since $\underline{u}=0$ is a subsolution to (3.1) while $\overline{u}=n$ is a supersolution, the result in [16] ensures the existence of a weak solution $u\in H^1(\Omega)$ which verifies $0\leq u\leq n$, and then $0\leq f(u)\leq f(n)$. In particular, $u\in L^\infty(\Omega)$, and by standard regularity theory $u\in C^{1,\alpha}(\overline{\Omega})$ (Corollary 8.35 in [30]). According to the comparison principle (see for instance Lemma 2.1 (ii) in [22]), this solution is unique.

In order to prove that the solution u to (3.1) so obtained solves $(P^n_\pm)$, we need to show that a suitable value of M can be selected. This will be accomplished by means of uniform bounds for the gradient. Indeed, if we consider $\hat{f}(t)=f(t)$ if $t\in[0,n]$ and $\hat{f}(t)=f(n)$ if $t>n$, it suffices with obtaining bounds only on $\partial\Omega$, as the following lemma shows.

Lemma 4. *Let u be the solution to* (3.1). *Then*

$$|\nabla u| \leq \sup_{\partial\Omega} |\nabla u|.$$

Proof. The proof relies on an application of the maximum principle to an equation satisfied by $|\nabla u|^2$. But we first regularize the problem: let $\{f_k\}_{k=1}^{\infty}, \{g_k\}_{k=1}^{\infty} \subseteq C^{\infty}(\mathbb{R})$ such that f_k and g_k are increasing for all k and $f_k \to \hat{f}$, $g_k \to g_M$ uniformly on compact sets of $\mathbb{R}$. We may moreover assume that the functions f_k, g_k are uniformly bounded. Consider the problems

$$\begin{cases} \Delta u \pm g_k(|\nabla u|^2) = f_k(u) & \text{in } \Omega, \\ u = n & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Arguing as before, there exists a unique solution $u_k \in C^{1,\alpha}(\overline{\Omega})$ to (3.2). Indeed, it follows by classical regularity that $u \in C^2(\overline{\Omega}) \cap C^{\infty}(\Omega)$. Let $v_k = |\nabla u_k|^2$. A calculation shows that

$$\Delta v_k = 2\sum_{i,j=1}^{N} (\partial_{ij} u_k)^2 + 2\sum_{i=1}^{N} \partial_i u_k \Delta(\partial_i u_k).$$

On the other hand, differentiating in (3.2) with respect to x_i, we obtain

$$\Delta(\partial_i u_k) \pm g_k'(|\nabla u_k|^2)\partial_i v_k = f_k'(u_k)\partial_i u_k,$$

so that

$$\Delta v_k \pm 2g_k'(v_k)\nabla u_k \nabla v_k = 2\sum_{i,j=1}^{N} (\partial_{ij} u_k)^2 + 2f'_k(u_k)v_k \geq 0 \quad \text{in } \Omega.$$

Thanks to the maximum principle, we have $v_k \leq \sup_{\partial\Omega} v_k$, that is,

$$|\nabla u_k|^2 \leq \sup_{\partial\Omega} |\nabla u_k|^2. \tag{3.3}$$

Our next purpose is passing to the limit in (3.3). Since f_k, g_k are uniformly bounded, we obtain that Δu_k is uniformly bounded in $\overline{\Omega}$, and thanks to Theorem 8.33 in [30],

$$|u_k|_{C^{1,\alpha}(\overline{\Omega})} \leq C$$

for some positive constant C, not depending on k, and some fixed $\alpha \in (0,1)$. Passing to a subsequence, we have $u_k \to u_0$ in $C^1(\overline{\Omega})$, and therefore u_0 is a weak solution to (3.1). By uniqueness, $u_0 = u$, and in particular $u_k \to u$ in $C^1(\overline{\Omega})$. We may pass to the limit in (3.3) to get the lemma proved. □

To obtain bounds for the gradient of u we need thus to estimate it on $\partial\Omega$. We will achieve this by constructing a suitable subsolution. Notice that, thanks to the regularity of Ω, it verifies a uniform exterior sphere condition, and hence there exists $R_1 > 0$ such that, for every $x_0 \in \partial\Omega$ we can find $z_0 \notin \overline{\Omega}$ with $\overline{B(z_0, R_1)} \cap \partial\Omega = \{x_0\}$. For $R_2 > R_1$ let A be the annulus $\{x \in \mathbb{R}^N : R_1 < |x - z_0| < R_2\}$.

Set $D = A \cap \Omega$ and denote $\Gamma_1 = \{x \in \mathbb{R}^N : |x - z_0| = R_1\}$, $\Gamma_2 = \{x \in \mathbb{R}^N : |x - z_0| = R_2\}$. Assume that the problem

$$\begin{cases} \Delta v \pm g(|\nabla v|) = f(v) & \text{in } A, \\ v = n & \text{on } \Gamma_1, \\ v = 0 & \text{on } \Gamma_2, \end{cases} \tag{3.4}$$

admits a radial subsolution $v \in C^1(\overline{A})$ and choose $M > \sup_{R_1 \le r \le R_2} v'(r)^2$. In that case, $g(|\nabla v|) = g_M(|\nabla v|^2)$, and since $v \le u$ on ∂D, we have by comparison $v \le u$ in D. Moreover, $v(x_0) = n = u(x_0)$, so that

$$\frac{\partial u}{\partial \nu}(x_0) \le \frac{\partial v}{\partial \nu}(x_0).$$

Now notice that $u = n$ on $\partial\Omega$ implies, by Hopf's principle, that $\frac{\partial u}{\partial \nu}(x_0) > 0$, and thus $\partial\Omega$ is a level set for u. This entails that ν is parallel to ∇u and hence $|\nabla u(x_0)| = \frac{\partial u}{\partial \nu}(x_0)$. It follows that

$$|\nabla u\,(x_0)|^2 \le v'(R_1)^2 < M.$$

In conclusion, u is a solution to $(P^n_\pm)$, as we wanted to see.

Thus the important point in the proof of existence of nonnegative solutions to $(P^n_\pm)$ is the obtention of a radial subsolution to (3.4), that is, a function $v \in C^1([R_1, R_2])$ verifying

$$(3.5)^\pm \qquad \begin{cases} v'' + \dfrac{N-1}{r} v' \pm g(|v'|) \ge f(v) & \text{in } R_1 < r < R_2, \\ v(R_1) = n, \\ v(R_2) = 0, \end{cases}$$

in the weak sense. For this aim, the cases with a plus sign and with a minus sign in $(3.5)^\pm$ have to be analyzed separately. It turns out that the problem with the $+$ sign is substantially simpler, since no further growth conditions on f nor g need to be imposed. On the other hand, the obtention of a subsolution for problem (3.5) with a minus sign is much more involved, and it strongly depends on the function g. In this case, we need to assume condition (1.8) since we will show in Section 5 that it is necessary for existence of solutions either to (P_-) or to (P^n_-) when n is large enough.

Proposition 5. *Let f, g be functions satisfying $(f_0 - g_0)$. Then*

i) (P^n_+) admits a unique nonnegative weak solution $u_n \in C^{1,\alpha}(\overline{\Omega})$ for every $n \in \mathbb{N}$.

ii) If (1.8) holds, (P^n_-) admits a unique nonnegative weak solution u_n for every $n \in \mathbb{N}$.

Moreover, in both cases one has that $0 < u_n < n$ in Ω and $u_n \in C^{1,\alpha}(\overline{\Omega})$ for every $\alpha \in (0,1)$.

Proof. We start proving *i)*. To obtain a subsolution to $(3.5)^+$, it suffices with choosing the unique nonnegative solution to

$$\begin{cases} \Delta v = f(v) & \text{in } A, \\ v = n & \text{on } \Gamma_1, \\ v = 0 & \text{on } \Gamma_2, \end{cases}$$

which is easily constructed by means of the method of sub and supersolutions, by taking $\underline{v} = 0$, $\overline{v} = n$. Thus the existence of a solution to (P^n_+) is obtained thanks to the previous discussion.

Now we prove *ii)*. As before, it suffices with constructing a radial function verifying $(3.5)^-$. Thus we need a function $v \in C^1([R_1, R_2])$ verifying

$$(3.6) \qquad \begin{cases} \left(r^{N-1} v'\right)' \ge r^{N-1} \left(f(v) + g(|v'|)\right), \\ v(R_1) = n, \\ v(R_2) = 0. \end{cases}$$

With the change of variables

$$s=\begin{cases}\log r & \text{if } N=2,\\ -\dfrac{1}{N-2}\dfrac{1}{r^{N-2}} & \text{if } N\geq 3,\end{cases}$$

and letting $w(s)=v(r)$, the inequality (3.6) gets transformed into:

$$\begin{cases} w''\geq h(s)\left(f(w)+g\left(\dfrac{1}{r^{N-1}}|w'|\right)\right),\\ w(a)=n,\\ w(b)=0,\end{cases} \tag{3.7}$$

where $a=\log R_1$, $b=\log R_2$ if $N=2$, while $a=-\frac{1}{N-2}\frac{1}{R_1^{N-2}}$, $b=-\frac{1}{N-2}\frac{1}{R_2^{N-2}}$ when $N\geq 3$. The function $h(s)=r^{2(N-1)}$ and $'$ stands now for differentiation with respect to s. To obtain (3.7), it is enough to have

$$\begin{cases} w''\geq R_2^{2(N-1)}\left(f(w)+g\left(R_1^{-(N-1)}|w'|\right)\right),\\ w(a)=n,\\ w(b)=0.\end{cases}$$

Our intention is keeping R_1 fixed (recall that it comes from the uniform exterior sphere condition) and treating R_2 as a parameter. Thus if $R_2\leq\bar{R}$, we are looking for a solution to

$$\begin{cases} w''=\bar{R}^{2(N-1)}\left(f(w)+g\left(R_1^{-(N-1)}|w'|\right)\right),\\ w(a)=n,\\ w(b)=0.\end{cases} \tag{3.8}$$

It is easily seen that solutions to (3.8) are decreasing. If we set $z(s)=w(b-s)$, then we look for an increasing function z which solves

$$\begin{cases} z''=\bar{R}^{2(N-1)}\left(f(z)+g\left(R_1^{-(N-1)}z'\right)\right),\\ z(0)=0,\\ z(b-a)=n.\end{cases}$$

It is therefore natural to analyze the initial value problem

$$\begin{cases} z''=\bar{R}^{2(N-1)}\left(f(z)+g\left(R_1^{-(N-1)}z'\right)\right),\\ z(0)=0,\\ z'(0)=z_0,\end{cases} \tag{3.9}$$

for $z_0>0$ and see if it is possible to choose z_0 so that $z(b-a)=n$. Since $b-a$ is expected to be small and n large enough, we should have in mind that z_0 is going to be chosen rather large.

For $z_0>0$, problem (3.9) admits a minimal solution, which is defined in an interval $[0,T)$ with $T\leq\infty$. Moreover, when $T<\infty$ we have $z(s)\to\infty$ or $z'(s)\to\infty$ as $s\to T-$. Notice that solutions are increasing and convex. They are also increasing with z_0.

We claim that condition (1.8) implies that $z(s)\to\infty$ as $s\to T$ (that is, the solution can not cease to exist because of the blow-up of the derivative). If this were not the case, we would have $z(s)\to\bar{z}$ as $s\to T$ for a certain finite $\bar{z}$ and $z'(s)\to\infty$ as $s\to T$. Let us rule out this possibility. If we multiply the equation by z', taking into account that z is bounded we have

$$\frac{z'z''}{A+B\,g\big(R_1^{-(N-1)}\,z'\big)}\le z',$$

for some positive constants A and B. Integrating in $[0,s)$ for s close to T and then letting $s\to T$, we obtain, with a change of variables in the integral:

$$\int_{R_1^{-(N-1)}z_0}^{\infty}\frac{\tau}{A+Bg(\tau)}\,d\tau\le R_1^{-(N-1)}\bar z,$$

which contradicts condition (1.8). Hence, we have shown that $T<\infty$ implies $z(s)\to\infty$ (and thus also $z'(s)\to\infty$) as $s\to T$.

Denote by $T(z_0)$ the maximal interval of existence of the minimal solution to (3.9). Two options may occur: either $T(z_0)=\infty$ for every $z_0>0$, that is, the minimal solution is always global, or there exists $z_1>0$ such that $T(z_1)<\infty$. In the first case, choosing $\delta>0$ small enough, and since $z(s)\ge z_0 s$ by convexity, we have $z(\delta)\ge z_0\delta>n$, provided z_0 is large enough. Hence there exists $s_0\in(0,\delta)$ such that $z(s_0)=n$. We may now choose R_2 such that $b-a=s_0$, and the existence of a function verifying (3.5) is shown in this case.

In the remaining case $T(z_1)<\infty$ for some $z_1>0$, we have, since solutions are increasing with z_0, that $T(z_0)<\infty$ for every $z_0>z_1$. In particular, for every $z_0>z_1$ there exists $s_0\in(0,T(z_0))$ such that $z(s_0)=n$. Since $z(s_0)\ge z_0s_0$, we also have $s_0\to 0$ as $z_0\to\infty$. We choose as before a large value of z_0 and then R_2 so that $b-a=s_0$, and in this way we have constructed the desired subsolution. This concludes the proof. □

4. Existence and nonexistence of solutions to $(P_\pm)$

This section is dedicated to prove Theorems 1 and 2, that is, existence and nonexistence results for problems $(P_\pm)$. Let us first comment on the method of proof. To show the existence of a solution to either problem, we consider the solutions u_n to the finite problems $(P_\pm^n)$ furnished by Proposition 5. If $B\subset\Omega$ is an arbitrary ball, then we obtain by the comparison principle in Lemma 2.1 (ii) of [22] that $u_n\le u_{n,B}$, where $u_{n,B}$ is the unique solution to $(P_\pm^n)$ in the ball B. Assume we prove

$$\sup_{n\in\mathbb{N}} u_{n,B}(x)<\infty \quad x\in B. \tag{4.1}$$

This would imply that u_n is locally uniformly bounded in Ω, and since the sequence u_n is increasing we have $u_n\to u:=\sup_{n\in\mathbb{N}}u_n$ point-wise in Ω. Thanks to Theorem A.1 in the Appendix, we would have that $|\nabla u_n|$ is locally uniformly bounded. Then we would obtain that $\Delta u_n=h_n$, for a function h_n which is locally uniformly bounded. Using classical regularity (for instance (4.45) in [30]), we obtain bounds for u_n in $C^{1,\alpha}_{\rm loc}(\Omega)$ for every $\alpha\in(0,1)$. It is then standard to conclude by means of a diagonal procedure that, passing to a subsequence, $u_n\to u$ in $C^1_{\rm loc}(\Omega)$. Then u is a weak solution to $(P_\pm)$, and by standard regularity $u\in C^{1,\alpha}_{\rm loc}(\Omega)$ for every $\alpha\in(0,1)$. We remark that in the previous reasoning the radius of the ball B can be taken as small as desired.

As for the nonexistence issue, let B be a large ball containing Ω and denote again by $u_{n,B}$ the solution to $(P_\pm^n)$ in B. If either problem (P_+) or (P_-) had a solution, we would obtain by comparison that $u\ge u_{n,B}$. We would arrive at a contradiction if we prove that

$$u_{n,B}\to\infty \quad \text{uniformly in } B. \tag{4.2}$$

Thus it is clear that only the radial case needs to be dealt with.

With these ideas in mind, we proceed to prove Theorems 1 and 2.

Proof of Theorem 1. We need only prove (4.1) or (4.2). For notational simplicity we will drop the subindex B and denote the solution to $(P_\pm^n)$ in the ball B by u_n. Notice that u_n has to be radially symmetric, and thus it verifies

$$\left\{ \begin{array}{l} u'' + \dfrac{N-1}{r} u' = f(u) - g(|u'|), \\ u'(0) = 0, \\ u(R) = n, \end{array} \right. \tag{4.3}$$

where $'$ stands for derivative with respect to $r = |x|$. Denote by $u_{0,n} = u_n(0) = \min u_n$.

Let us prove part (i), that is, nonexistence of solutions for (P_+). Assume first f does not verify the Keller-Osserman condition (1.2). Since, according to Proposition 3, $u' > 0$ we have from (4.3) that $u'' \le f(u)$, so that multiplying by u' and integrating in $(0, R)$ we arrive at $u' \le \sqrt{2F(u)}$, where F is the primitive of f vanishing at zero. Thus

$$\int_{u_{0,n}}^{n} \frac{ds}{\sqrt{2F(s)}} \le R.$$

Letting $n \to \infty$ we obtain that $u_{0,n} \to \infty$ since f does not verify (1.2). Thus (4.2) holds and this shows nonexistence.

Next, suppose condition (1.6) holds. Thanks to Proposition 3, we have $u', u'' \ge 0$, so that (4.3) implies $f(u) \ge g(u')$. Hence

$$\frac{u'}{g^{-1}\big(f(u)\big)} \le 1,$$

and integrating in $(0, R)$:

$$\int_{u_{0,n}}^{n} \frac{ds}{g^{-1}\big(f(s)\big)} \le R.$$

Letting $n \to \infty$ and using condition (1.7) we deduce again that $u_{0,n} \to \infty$, and hence no nonnegative solutions to (P_+) exist.

Let us prove now the existence result in part (ii). We first show that condition (1.7) allows us to construct a radial supersolution in a ball B which blows up on ∂B, provided that the radius R of the ball is small enough. We will assume for the moment that $R \le 1/2$, and will search for a supersolution in the form

$$\bar{u}(r) = \phi(R^2 - r^2),$$

where $\phi(0) = \infty$. It is not hard to show that $\bar{u}$ will be a supersolution if

$$4r^2 \phi'' - 2N\phi' + g\big(2r|\phi'|\big) \le f\big(\phi\big),$$

where $'$ stands for differentiation with respect to $t = R^2 - r^2$. Assume for the moment that $\phi' < 0$, $\phi'' > 0$. Then it suffices to have

$$\phi'' - 2N\phi' + g(-\phi') \le f(\phi). \tag{4.4}$$

Let us now choose the function ϕ. Thanks to condition (1.7), the problem

$$\left\{ \begin{array}{l} \Gamma(|\phi'|) = \dfrac{1}{2} F(\phi) \quad t > 0, \\ \phi(0) = \infty, \end{array} \right. \tag{4.5}$$

admits a unique solution. It is more or less standard tho check that $\phi' < 0$ (and moreover $\phi(t)$, $\phi'(t) \to 0$ as $t \to \infty$). Notice that the convergence of the integral in (1.7) implies

$$\frac{\Gamma^{-1}\Big(\frac{1}{2}F(s)\Big)}{s} \to \infty \qquad (s \to \infty),$$

and hence

$$\frac{|\phi'(t)|}{|\phi(t)|} \to \infty \qquad (t \to 0).$$

In particular, there exists $\varepsilon > 0$ such that $\phi(t) \le |\phi'(t)|$ if $0 < t \le \varepsilon$. Restrict R further to have $R^2 \le \varepsilon$, so that $t = R^2 - r^2 \le \varepsilon$. Let us see that ϕ verifies the required properties. Taking derivative in (4.5) we have

$$\phi'' = \frac{1}{2}\, f(\phi)\, \frac{-\phi'}{2g(-2\phi') + 4N(-\phi')}.$$

We deduce then that $\phi'' > 0$ and

$$\phi'' \le \frac{1}{2}\, f(\phi). \tag{4.6}$$

On the other hand, we have by the monotonicity of f and g:

$$F(t) = \int_0^t f(s)\, ds \le f(t)t$$

and

$$\Gamma(t) = \int_0^{2t} g(s)ds + 2Nt^2 \ge \int_t^{2t} g(s)ds + 2Nt^2 \ge tg(t) + 2Nt^2.$$

Then

$$g(-\phi') - 2N\phi' \le \frac{\Gamma(-\phi')}{-\phi'} = \frac{1}{2}\frac{F(\phi)}{-\phi'} \le \frac{1}{2}\frac{f(\phi)\phi}{-\phi'} \le \frac{1}{2} f(\phi). \tag{4.7}$$

Adding (4.6) and (4.7) we obtain (4.4).

To summarize, we have constructed a supersolution $\bar{u}$ to (4.3) in B, with $\bar{u} = \infty$ on ∂B, with the only restriction that R is small enough. It follows by comparison that $u_{n,B} \le \bar{u}$ in B, and hence (4.1) follows, provided only that R is sufficiently small. Thus existence of a nonnegative solution to (P_+) is proved in this case. □

Proof of Theorem 2. To show part (i), we prove (4.2). Recall that, the solution $u_{n,B}$ to $(P^n_\pm)$ verifies the Cauchy problem

$$\begin{cases} u'' + \dfrac{N-1}{r}\, u' = f(u) + g(|u'|), \\ u'(0) = 0, \\ u(0) = u_{0,n}, \end{cases} \tag{4.8}$$

and thanks to Proposition 3, it verifies $u'(r) \ge 0$, $u(r) \le u_0 + Ru'(r)$. Hence if we assume that $R \ge 1$, we also have $u'' \le f(u) + g(u') \le f(u_{0,n} + Ru') + g(u') \le f(u_{0,n} + Ru') + g(u_{0,n} + Ru')$. It follows after an integration in $(0, R)$ that

$$\int_0^{u'(R)} \frac{ds}{f\big(u_{0,n} + R\, s\big) + g\big(u_{0,n} + R\, s\big)} \le R.$$

Let us set $u_{0,n} + Rs = \tau$; since the solution to $(P^n_\pm)$ verifies $n = u(R) \le u_{0,n} + Ru'(R)$, we arrive at

$$\int_{u_{0,n}}^{n} \frac{d\tau}{f(\tau)+g(\tau)} \le 1.$$

We mention in passing that $u_{0,n} = 0$ is only possible when the integral converges at zero, so no problem arises in this case (cf. a related situation in Theorem 1.2 in [21]). The divergence of the integral implies $u_{0,n} \to \infty$ as $n \to \infty$, and (4.2) gets proved. Thus no solutions exist under condition (1.9).

To show the existence result in part (ii), we argue as in Theorem 1 and prove (4.1). Notice that, if f verifies the Keller-Osserman condition (1.2), and since $u_{n,B}$ satisfies $\Delta u \ge f(u)$ in B, we obtain by comparison that $u_{n,B} \le U$, where U is the minimal solution to $\Delta U = f(U)$ in B, with $U = \infty$ on ∂B. Thus (4.1) is immediate in this case.

So assume $\lim_{t\to\infty} f(t) = \infty$ and g verifies condition (1.10). Since the equation in (4.8) can be written as $(r^{N-1}u')' = r^{N-1}\big(f(u)+g(u')\big)$, we can integrate in $(0,r)$ for an arbitrary r and use that u and u' are increasing (Proposition 3) to obtain

$$\begin{aligned} u'(r) &= \frac{1}{r^{N-1}} \int_0^r s^{N-1}\Big(f\big(u(s)\big)+g\big(u'(s)\big)\Big)\,ds \\ &\le \frac{r}{N}\Big(f\big(u(r)\big)+g\big(u'(r)\big)\Big). \end{aligned}$$

Taking this inequality to (4.8) we obtain that

$$u'' \ge \frac{1}{N}\big(f(u)+g(u')\big) \ge \frac{1}{N}\big(f(u_{0,n})+g(u')\big). \tag{4.9}$$

When $u_{0,n} > 0$, it follows by integrating in $(0,R)$ and performing the standard change of variables $s = u'(r)$ in the integral that

$$\frac{1}{N}\,R \le \int_0^{u'(R)} \frac{1}{f(u_{0,n})+g(s)}\,ds \le \int_0^{\infty} \frac{1}{f(u_{0,n})+g(s)}\,ds. \tag{4.10}$$

If we had $u_{0,n} \to \infty$ the last integral in (4.10) would tend to zero and we would reach a contradiction. To see this, take an arbitrary $\varepsilon > 0$, and let $M > 0$ be such that

$$\int_M^{\infty} \frac{ds}{g(s)} \le \varepsilon.$$

Then

$$\frac{1}{N}\,R \le \int_0^M \frac{ds}{f(u_{0,n})} + \int_M^\infty \frac{ds}{g(s)} \le \frac{M}{f(u_{0,n})} + \varepsilon.$$

Letting $n \to \infty$ and then $\varepsilon \to 0$, we have the contradiction. Thus $u_{0,n}$ remains bounded and this shows (4.1), as was to be proved. □

5. Necessity of condition (1.8)

In this section we will prove that condition (1.8) is necessary for problem (P_-) to have a solution. Moreover, it is also necessary for problem (P^n_-) when n is sufficiently large. This result is in the spirit of the nonexistence example in [51] (see Theorem 1 in Chapter III, §16 there).

Thus our main result here is:

Theorem 6. *Let f and g be functions satisfying $(f_0 - g_0)$, and assume that $\lim_{t\to\infty} f(t) = \infty$. If condition* (1.8) *does not hold, that is,*

$$\int_1^\infty \frac{s}{g(s)}\, ds < \infty, \tag{5.1}$$

then problem (P_-) does not have any nonnegative solution. Moreover, there exists $n_0 = n_0(\Omega)$ such that the problem with finite boundary datum (P_-^n) does not have nonnegative solutions $u \in C^1(\overline{\Omega})$ for $n \geq n_0$.

Remark 3. A slight modification of the proof of Theorem 6 shows that if $h \in C^1(\partial\Omega)$ is positive, the problem

$$\begin{cases} \Delta u - g(|\nabla u|) = f(u) & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega, \end{cases}$$

has no nonnegative solutions $u \in C^1(\overline{\Omega})$ if $|h|_\infty$ is large enough.

Let us begin by considering the particular instance where $\Omega = B_R$. In this case, we will show that no radial solutions exist in both situations.

Lemma 7. *Assume g verifies* (5.1)*. Then problem (P_-) does not have radial nonnegative solutions. Moreover, the problem with finite datum*

$$\begin{cases} \Delta u - g(|\nabla u|) = f(u) & \text{in } B_R, \\ u = n & \text{on } \partial B_R, \end{cases} \tag{5.2}$$

does not have nonnegative solutions if n is large.

Proof. Let us show first that (P_-) does not admit radial nonnegative solutions. It follows from (4.9) in the proof of Theorem 2 that

$$u'u'' \geq \frac{1}{N}\big(f(u) + g(u')\big)u' \geq \frac{1}{N} g(u')u'.$$

Thus if we divide by $g(u')$ and integrate between r_0 and r for some arbitrary $r_0 \in (0, R)$ and r close to R, we obtain

$$\int_{u'(r_0)}^{u'(r)} \frac{s}{g(s)} ds \geq u(r) - u(r_0),$$

and we arrive at a contradiction when we let $r \to R$ thanks to (5.1). Hence no radial nonnegative solutions to (P_-) can exist.

The proof that no solutions to (5.2) exist when n is large enough is a bit more involved. We assume that there exists a sequence $n_k \to \infty$ such that each problem (5.2) with $n = n_k$ has a nonnegative solution u_k. By uniqueness, this solution must be radial, and hence it verifies

$$\begin{cases} u'' + \dfrac{N-1}{r} u' = f(u) + g(|u'|), \\ u'(0) = 0, \\ u(R) = n_k. \end{cases}$$

Let $u_{0,k} = u_k(0)$. Since we may assume that n_k is increasing in k and solutions to the associated initial value problem are also increasing with respect to the initial datum, we have that $u_{0,k}$ is increasing. If $u_{0,k} \to \infty$, we would obtain (4.10), leading to a contradiction as in the proof of Theorem 2. Thus $u_{0,k}$ is bounded and we have $u_{0,k} \to u_0$ for some $u_0 \geq 0$. We will assume $u_0 > 0$, since with a slight modification of the arguments below the case $u_0 = 0$ can also be covered.

Let u be a solution to the Cauchy problem

$$\begin{cases} u'' + \dfrac{N-1}{r} u' = f(u) + g(u'), \\ u(0) = u_0, \\ u'(0) = 0, \end{cases} \tag{5.3}$$

defined in the maximal interval of existence $[0,T)$. It is possible that $T = \infty$, but if $T < \infty$ then

$$\lim_{r\to T} u'(r) = \infty,$$

while

$$\lim_{r\to T} u(r) < \infty,$$

since by the first part of the proof there are no solutions which blow-up at any finite T.

On the other hand, the derivatives are also ordered, so that for small $\varepsilon > 0$, $u_k(r) \le u(r)$, $u_k'(r) \le u'(r)$ if $r \in [0, \min\{T,R\}-\varepsilon]$. With an argument like the one used at the beginning of Section 4, we obtain that u_k converges in $C^1[0,\min\{T,R\}-\varepsilon]$ to a function u which has to verify problem (5.3). We warn the reader that this function u does not necessarily coincide with the one introduced before, since uniqueness does not necessarily hold for (5.3).

We claim that $T < R$. Indeed, if we had $T \ge R$, since $u_k(R) = n_k \le u(R)$, we would arrive at $u(R) = \infty$, which is not possible. Thus $T < R$. Since for small $\varepsilon > 0$ we have $u_k' \to u'$ uniformly in $[0, T-\varepsilon]$ we have $u_k'(T-\varepsilon) \ge u'(T-\varepsilon) - \varepsilon$ if k is large enough. Using again that $u_k'' \ge \frac{1}{N} g(u_k')$ we have, integrating in $[T-\varepsilon, R]$, that

$$\frac{1}{N}(R - T + \varepsilon) \le \int_{u_k'(T-\varepsilon)}^{u_k'(R)} \frac{ds}{g(s)}.$$

Hence

$$\frac{1}{N}(R - T + \varepsilon) \le \int_{u'(T-\varepsilon)-\varepsilon}^{\infty} \frac{ds}{g(s)},$$

and letting $\varepsilon \to 0$ we arrive at $T \ge R$, a contradiction. Thus there can be no solutions to (5.2) when n is large enough, and this concludes the proof. □

Let us now prove Theorem 6. The proof relies in showing that if a solution to either (P_-) or to $(P_\pm^n)$ exists in a smooth bounded domain Ω, then we can construct a solution for the same problem in a suitable ball B_R, contradicting Lemma 7. The argument to do this is similar to the one used in Section 3: we truncate the function g, find a solution to the truncated problem and then show that it is a solution to the original problem by obtaining bounds for the gradient.

Proof of Theorem 6. Assume first that for some n there exists a solution $u \in C^1(\overline{\Omega})$ to

$$\begin{cases} \Delta u - g(|\nabla u|) = f(u) & \text{in } \Omega, \\ u = n & \text{on } \partial\Omega. \end{cases} \tag{5.4}$$

Take an arbitrary ball $B_R \subset \Omega$ which is tangent to $\partial\Omega$ at some $x_0 \in \partial\Omega$. For $M > \sup_{\overline{\Omega}} |\nabla u|^2$, consider the truncated problem

$$\begin{cases} \Delta v - g_M(|\nabla v|^2) = f(u) & \text{in } B_R, \\ v = n & \text{on } \partial B_R, \end{cases}$$

where g_M is a bounded function with $g_M(t^2) = g(t)$ if $t^2 \le M$. As in Section 3, there exists a nonnegative solution v to this problem by means of the method of sub and supersolutions.

By uniqueness this solution is radial and by Proposition 3, it verifies $v' > 0$, while v' is increasing.

On the other hand, we have by comparison that $u < v$ in B_R, and since $u(x_0) = v(x_0)$:

$$\frac{\partial v}{\partial \nu}(x_0) \le \frac{\partial u}{\partial \nu}(x_0),$$

so that

$$v'(R)^2 \le |\nabla u(x_0)|^2 < M.$$

Since v' is increasing and positive we have $v'(r)^2 < M$ for $0 \le r \le R$, so that v is a solution to (5.2), contradicting Lemma 7 if n is large enough (depending only on R).

Finally, let us tackle the question of nonexistence for problem (P_-). Assume there exists a solution u to (P_-). Take a ball $B_R(x_0) \subset \Omega$ tangent to $\partial\Omega$ at some point z_0. For small $\varepsilon > 0$ let $x_\varepsilon = x_0 - \varepsilon\nu(z_0)$, where $\nu(z_0)$ is the outward unit normal to $\partial\Omega$ at z_0, so that $B_R(x_\varepsilon) \subset\subset \Omega$. Denote $n = n(\varepsilon) = \sup_{\partial B_R(x_\varepsilon)} u$. Then

$$\begin{cases} \Delta u - g(|\nabla u|) = f(u) & \text{in } B_R(x_\varepsilon), \\ u \le n & \text{on } \partial B_R(x_\varepsilon), \end{cases}$$

and an argument like in the first part of the proof shows that the problem

$$\begin{cases} \Delta v - g(|\nabla v|) = f(v) & \text{in } B_R(x_\varepsilon), \\ v = n & \text{on } \partial B_R(x_\varepsilon), \end{cases}$$

has a solution. However, when $\varepsilon \to 0$, $n \to \infty$, and we arrive at a contradiction with the already proved nonexistence of solutions to (5.4) for large n. This concludes the proof. $\square$

6. Uniqueness

In this section we consider some results concerning uniqueness of nonnegative solutions to $(P_\pm)$. As we mentioned earlier, uniqueness is achieved thanks to some special monotonicity of the nonlinearities f and g, together with the knowledge of the exact boundary behavior of all possible nonnegative solutions. To obtain this boundary behavior, we will assume that f verifies the Keller-Osserman condition (1.2), and another condition which is usual in the literature, namely:

$$\limsup_{t\to\infty} \frac{\psi_f(\lambda t)}{\psi_f(t)} < 1 \tag{6.1}$$

for every $\lambda > 1$, where

$$\psi_f = \int_t^\infty \frac{ds}{\sqrt{F(s)}} \tag{6.2}$$

(cf. [7]). Let us mention that this condition holds for instance when $f(t)/t^p$ is increasing for some p and large t, as can be easily checked.

We prove the following theorems:

Theorem 8. *Let f and g be functions satisfying $(f_0 - g_0)$ and assume that* (6.1) *holds, $f(t)/t$ is increasing and $g(t)/t$ is decreasing. Then problem (P_+) admits a unique nonnegative solution.*

Theorem 9. *Let f and g be functions satisfying $(f_0 - g_0)$. Assume that g verifies* (1.8), (1.10) *and* (6.1) *holds with ψ_f replaced by ψ_g, where*

$$\psi_g(t) = \int_t^\infty \frac{ds}{g(s)}.$$

Assume moreover that $f(t)/t$ and $g(t)/t$ are increasing and that

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = 0.$$

Then problem (P_-) has a unique nonnegative solution $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$.

Let us begin with problem (P_+). The boundary behavior is given in the next lemma, where we denote $d(x) = \mathrm{dist}(x, \partial\Omega)$.

Lemma 10. *Assume f and g are increasing continuous functions with $f(0) = g(0) = 0$, and that f verifies the Keller-Osserman condition* (1.2)*. Assume moreover that g verifies*

$$\limsup_{t\to\infty} \frac{g(t)}{t} < \infty. \tag{6.3}$$

Then every nonnegative solution u to (P_+) verifies

$$\lim_{d(x)\to 0} \frac{\psi_f\big(u(x)\big)}{d(x)} = 1, \tag{6.4}$$

where ψ_f is given by (6.2)*. If in addition f verifies* (6.1)*, then*

$$\lim_{d(x)\to 0} \frac{u(x)}{\phi_f\big(d(x)\big)} = 1, \tag{6.5}$$

where ϕ_f is the inverse function of ψ_f.

Proof. The proof of boundary estimates relies on comparison with solutions in balls and annuli. For this sake, we need to analyze first the radial case. Thus let u be a solution to

$$\begin{cases} u'' + \dfrac{N-1}{r}\, u' = f(u) - g(|u'|), \\ u'(0) = 0, \\ u(R) = \infty. \end{cases}$$

It follows from Proposition 3 that $u'' \geq 0$, $u' > 0$, so that $u'' \leq f(u)$. Multiplying by u' and integrating we obtain $u' \leq \sqrt{2F(u)}$, so that in a standard way:

$$\int_{u(r)}^{\infty} \frac{ds}{\sqrt{2F(s)}} \leq R - r,$$

that is,

$$\limsup_{r\to R} \frac{\psi_f\big(u(r)\big)}{R-r} \leq 1.$$

To obtain the complementary inequality, we first observe that since f verifies the Keller-Osserman condition, it follows that

$$\lim_{t\to\infty} \frac{\sqrt{F(t)}}{f(t)} = 0$$

(see the Appendix in [6]) and thus condition (6.3) implies that for some positive constant C:

$$\frac{g(u'(r))}{f(u(r))} \leq C \frac{u'(r)}{f(u(r))} \leq C \frac{\sqrt{2F(u(r))}}{f(u(r))} \to 0 \quad \text{as } r \to R.$$

Now notice that $\frac{u'(r)}{r} \le \frac{\sqrt{2F(u)}}{r_0}$ if $r \ge r_0$, where $r_0 \in (0,R)$ is arbitrary. Thus

$$\begin{aligned} u'' &\ge f(u) - g(u') - \frac{N-1}{r_0}\sqrt{2F(u)} \\ &\ge f(u)\left(1 - \frac{g(u')}{f(u)} - \frac{N-1}{r_0}\frac{\sqrt{2F(u)}}{f(u)}\right) \\ &\ge (1-\varepsilon)f(u) \end{aligned}$$

for some small ε, provided r is close enough to R. An integration as before provides

$$\liminf_{r\to R} \frac{\psi_f\big(u(r)\big)}{R-r} \ge 1.$$

Thus (6.4) holds in this case. It is shown with only minor modifications that it also holds when the domain Ω is an annulus.

Now let u be a solution to (P_+) in a smooth bounded domain Ω. Choose a radius $R>0$ such that $\partial\Omega$ verifies the uniform interior sphere condition. For $x \in \Omega$ with $d(x) < R$, let $\bar{x} \in \partial\Omega$ be its projection onto $\partial\Omega$. There exists $z_{\bar{x}} \in \Omega$ such that the ball $B_R(z_{\bar{x}})$ is contained in Ω and is tangent to $\partial\Omega$ at $\bar{x}$. Let u_B be a solution to (P_+) in this ball. We obtain by comparison:

$$u(x) \le u_B(|x - z_{\bar{x}}|),$$

for $x \in B_R(z_{\bar{x}})$. Using $d(x) = |x-\bar{x}| = R - |x - z_{\bar{x}}|$, and since ψ_f is decreasing,

$$\frac{\psi_f\big(u(x)\big)}{d(x)} \ge \frac{\psi_f\big(u_B(|x-z_{\bar{x}}|)\big)}{d(x)} = \frac{\psi_f\big(u_B(|x-z_{\bar{x}}|)\big)}{R-|x-z_{\bar{x}}|}. \tag{6.6}$$

Thanks to (6.4) in the radial case, we know that for a small $\varepsilon > 0$ there exists $\delta > 0$ such that for $d(x) < \delta$ the last term in (6.6) is $\ge 1-\varepsilon$. Hence

$$\frac{\psi_f\big(u(x)\big)}{d(x)} \ge 1-\varepsilon \quad \text{when } d(x) < \delta.$$

This immediately gives

$$\liminf_{d(x)\to 0} \frac{\psi_f\big(u(x)\big)}{d(x)} \ge 1.$$

For the complementary inequality we use the uniform exterior sphere condition. Given $x \in \Omega$ close to $\partial\Omega$, we take its projection $\bar{x} \in \partial\Omega$. There exist $R' > 0$ (not depending on x) and $w_{\bar{x}} \notin \overline{\Omega}$ such that $B_{R'}(w_{\bar{x}}) \cap \Omega = \varnothing$, $\overline{B_{R'}(w_{\bar{x}})} \cap \overline{\Omega} = \{\bar{x}\}$. Take $R'' \gg 1$ so that $\Omega \subset B_{R''}(w_{\bar{x}})$.

Then $\Omega \subseteq A := B_{R''}(w_{\bar{x}}) \setminus \overline{B_{R'}(w_{\bar{x}})}$ and taking any nonnegative solution u_A to (P_+) in the annulus A we obtain by comparison $u(x) \ge u_A(|x - w_{\bar{x}}|)$, so that

$$\frac{\psi_f\big(u(x)\big)}{d(x)} \le \frac{\psi_f\big(u_A(|x-W_{\bar{x}}|)\big)}{|x-W_{\bar{x}}| - R'} \le 1+\varepsilon,$$

when $d(x) < \delta$. Thus (6.4) is proved.

Finally, when condition (6.1) holds, it is well known that (6.4) implies (6.5). However, we will include a proof for completeness.

For fixed $\lambda > 1$ and fixed positive ε we have $\psi_f(\lambda t) \le (1-\varepsilon)\psi(t)$ for large enough t, hence $\lambda\phi_f(s) \ge \phi_f((1-\varepsilon)s)$ for small enough s. Since $\lambda > 1$ is arbitrary this yields

$$\limsup_{s\to 0} \frac{\phi_f((1-\varepsilon)s)}{\phi_f(s)} \le 1. \tag{6.7}$$

On the other hand, thanks to (6.4), there exists $\delta > 0$ such that for $d(x) < \delta$ we have $\psi_f(u(x)) \geq (1-\varepsilon)d(x)$. This implies

$$\frac{u(x)}{\phi_f(d(x))} \leq \frac{\phi_f((1-\varepsilon)d(x))}{\phi_f(d(x))}.$$

Finally, the inequality (6.7) implies

$$\limsup_{d(x)\to 0} \frac{u(x)}{\phi_f(d(x))} \leq 1,$$

and the lower inequality for the inferior limit in (6.5) is shown similarly. This concludes the proof. □

We now prove the uniqueness result for problem (P_+). The proof is an adaptation of the argument in [26].

Proof of Theorem 8. Let u, v be arbitrary nonnegative solutions to (P_+). Notice that under the assumptions on g we have (6.3), so that Lemma 10 can be applied and it gives

$$\lim_{d(x)\to 0} \frac{u(x)}{v(x)} = 1.$$

Choose $\varepsilon > 0$. Then there exists $\delta > 0$ so that

$$(1-\varepsilon)v(x) \leq u(x) \leq (1+\varepsilon)v(x) \tag{6.8}$$

for $d(x) \leq \delta$. Let $\Omega^\delta = \{x \in \Omega : d(x) > \delta\}$ and consider the problem

$$\begin{cases} \Delta w + g(|\nabla w|) = f(w) & \text{in } \Omega^\delta, \\ w = u & \text{on } \partial\Omega^\delta, \end{cases} \tag{6.9}$$

which has as its unique solution $w = u$. Now the monotonicity of $f(t)/t$ and $g(t)/t$ implies that $(1+\varepsilon)v$ is a supersolution to (6.9), while $(1-\varepsilon)\,v$ is a subsolution. It follows by comparison that $(1-\varepsilon)v \leq u \leq (1+\varepsilon)v$ in Ω^δ. Hence the inequality (6.8) holds throughout Ω, and we can let $\varepsilon \to 0$ to obtain $u = v$. This shows uniqueness. □

Now let us turn once again to problem (P_-). We have an analogue of Lemma 10:

Lemma 11. *Assume g is an increasing continuous function which verifies $g(0) = 0$, (1.10) and*

$$\int_1^\infty \frac{s}{g(s)}ds = \infty,$$

and (6.1) holds with ψ_f replaced by ψ_g, where

$$\psi_g(t) = \int_t^\infty \frac{ds}{g(s)}.$$

If f is an increasing continuous function with $f(0) = 0$ and

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = 0, \tag{6.10}$$

then every nonnegative solution to (P_-) verifies

$$u(x) \sim \int_{d(x)}^1 \phi_g(t)dt \tag{6.11}$$

as $d(x) \to 0$, where ϕ_g is the inverse function of ψ_g.

Proof. The proof of (6.11) is obtained by comparison with solutions in balls and annuli, as in Lemma 10. Thus we only prove it in the case $\Omega = B_R$. If u is a radial nonnegative solution, then

$$u'' + \frac{N-1}{r}u' = f(u) + g(u') \geq g(u'). \tag{6.12}$$

Notice that u'' is not necessarily increasing, but the group $u'' + \frac{N-1}{r}u'$ is, thanks to Proposition 3. Thus for every $r_0 \in (0,R)$ and $r \in (r_0,R)$:

$$\begin{aligned} u'(r) = u'(r_0) + \int_{r_0}^r u''(s) &\leq u'(r_0) + \int_{r_0}^r \left(u''(s) + \frac{N-1}{s}u'(s)\right) ds \\ &\leq u'(r_0) + (r-r_0)\left(u''(r) + \frac{N-1}{r}u'(r)\right) \\ &\leq u'(r_0) + (N-1)\frac{R-r_0}{r_0}u'(r) + (R-r_0)u''(r). \end{aligned}$$

Taking r_0 close enough to R so that $(N-1)\frac{R-r_0}{r_0} \leq \frac{1}{2}$, we obtain

$$u'(r) \leq 2u'(r_0) + 2(R-r_0)u''(r).$$

Dividing by $u''(r)$, letting $r \to R$ and then $r_0 \to R$, we obtain $u'(r)/u''(r) \to 0$ as $r \to R$. Thus from (6.12):

$$(1+\varepsilon)u'' \geq g(u'),$$

when r is close enough to R. Dividing by $g(u')$ and integrating in (r,R) for r close to R:

$$\int_{u'(r)}^\infty \frac{ds}{g(s)} \geq \frac{1}{1+\varepsilon}(R-r).$$

Hence we arrive at

$$\liminf_{r\to R} \frac{\psi_g\big(u'(r)\big)}{R-r} \geq 1.$$

On the other hand, since u' is increasing, it is shown as before that $u(r)/u'(r) \to 0$ as $r \to R$, and in particular $u \leq u'$ if r is close to R. Thanks to condition (6.10) we also have $f(t) \leq \varepsilon g(t)$ when t is large enough. Then

$$u'' \leq u'' + \frac{N-1}{r}\,u' = f(u) + g(u') \leq f(u') + g(u') \leq (1+\varepsilon)g(u'),$$

and it follows that

$$\int_{u'(r)}^\infty \frac{ds}{g(s)} \leq (1+\varepsilon)(R-r).$$

We then obtain

$$\lim_{r\to R} \frac{\psi_g\big(u'(r)\big)}{R-r} = 1.$$

With the additional condition (6.1) we also have

$$\lim_{r\to R} \frac{u'(r)}{\phi_g(R-r)} = 1,$$

as in Lemma 10. Thanks to l'Hôpital rule we deduce that

$$u(r) \sim u(0) + \int_0^r \phi_g(R-s)ds = u(0) + \int_{R-r}^R \phi_g(t)dt \sim \int_{R-r}^1 \phi_g(t)dt,$$

which is (6.11) in a ball of radius R. This concludes the proof. □

Remark 4. We notice that $\phi_g' = -g(\phi_g)$, so that

$$\int_{R-r}^{1} \phi_g(t)dt = -\int_{R-r}^{1} \frac{\phi_g(t)\phi_g'(t)}{g\big(\phi_g(t)\big)}dt = \int_{\phi_g(1)}^{\phi_g(R-r)} \frac{s}{g(s)}ds,$$

and the boundary behavior (6.11) can be written as

$$u(x) \sim \int_{1}^{\phi_g(d(x))} \frac{s}{g(s)}ds$$

when $d(x) \to 0$.

We close this section by mentioning that the proof of Theorem 9 is a slight variation of that of Theorem 8, using Lemma 11 instead of Lemma 10, and therefore it will not be given.

7. Some examples

In this section we quote some important cases of nonlinearities f and g, and particularize the results of our paper to them. It is easily seen that conditions (a) or (b) in the Introduction hold for all of them. Let us begin with (P_+).

7.1. (P_+) with $g(t) = t^q$ for some $q > 0$. In this case $g^{-1}(t) = t^{1/q}$, while $\Gamma(s) = \frac{(2s)^{q+1}}{q+1} + 2Ns^2$. Thus when $0 < q \le 1$ we have $\Gamma(s) \sim \text{constant} \cdot s^2$ as $s \to \infty$, so that $\Gamma^{-1}(s) \sim \text{constant} \cdot \sqrt{s}$ as $s \to \infty$, and condition (1.7) is nothing more than Keller-Osserman condition (1.2). When $q > 1$, on the contrary, $\Gamma(s) \sim \text{constant} \cdot s^{q+1}$ as $s \to \infty$, so that $\Gamma^{-1}(s) \sim \text{constant} \cdot s^{\frac{1}{q+1}}$ and (1.7) reads as

$$\int_{1}^{\infty} \frac{ds}{F(s)^{\frac{1}{q+1}}} < \infty.$$

So that we obtain directly from Theorem 1:

Corollary 12. *Let f be an increasing continuous function with $f(0) = 0$. Then for $0 < q \le 1$, the problem*

$$\begin{cases} \Delta u + |\nabla u|^q = f(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{7.1}$$

admits a nonnegative solution if and only if f verifies the Keller-Osserman condition (1.2), *while for $q > 1$,* (7.1) *admits a nonnegative solution when*

$$\int_{1}^{\infty} \frac{ds}{F(s)^{\frac{1}{q+1}}} < \infty \tag{7.2}$$

and does not have any nonnegative solution when

$$\int_{1}^{\infty} \frac{ds}{f(s)^{\frac{1}{q}}} = \infty. \tag{7.3}$$

Let us mention that conditions (7.2) and (7.3) are not exhaustive. This is easily seen by taking $f(t) = t^q(\log t)^\alpha$, for α verifying $q < \alpha \le q+1$, where neither condition (7.2) nor (7.3) hold.

As particular cases of Corollary 12, let us single out the functions $f(t) = t^p$, $p > 0$ and $f(t) = e^t - 1$, where all possibilities are exhausted. The next two corollaries complement the results in [4], since uniqueness and nonexistence were not considered there.

Corollary 13. *Let p, $q > 0$. For the problem*

$$\begin{cases} \Delta u + |\nabla u|^q = u^p & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases}$$

we have:

(i) *If $0 < q \le 1$, there exists a nonnegative solution if and only if $p > 1$, and it is unique.*
(ii) *If $q > 1$, there exists a nonnegative solution if and only if $p > q$.*

Corollary 14. *The problem*

$$\begin{cases} \Delta u + |\nabla u|^q = e^u - 1 & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases}$$

admits a nonnegative solution for every $q > 0$. This solution is unique when $0 < q \le 1$.

7.2. (P_+) with $g(t) = e^t - 1$. This is an interesting example since we are analyzing a problem with a huge growth in the gradient. Notice that $\Gamma(s) \sim e^{2s}$ for large s so that $\Gamma^{-1}(s) \sim \frac{1}{2}\log s$ as $s \to \infty$. Since also $g^{-1}(t) \sim \log t$ we immediately have from Theorem 1 that the condition

$$\int_1^\infty \frac{ds}{\log F(s)} < \infty \tag{7.4}$$

implies existence, while

$$\int_1^\infty \frac{ds}{\log f(s)} = \infty \tag{7.5}$$

gives nonexistence. The important remark is that conditions (7.4) and (7.5) are complementary. This is easily seen by noticing that $F(2t) \ge f(t)$ for $t \ge 1$, so that the divergence of the integral in (7.4) implies (7.5) and the convergence of the integral in (7.5) implies condition (7.4). Thus:

Corollary 15. *Let f a continuous increasing function verifying $f(0) = 0$. Then the problem*

$$\begin{cases} \Delta u + e^{|\nabla u|} - 1 = f(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases}$$

admits a nonnegative solution if and only if

$$\int_1^\infty \frac{ds}{\log f(s)} < \infty.$$

Let us finally consider problem (P_-).

7.3. (P_-) with $g(t) = t^q$ for some $q > 0$. Condition (1.8) is equivalent to $q \le 2$, while (1.10) in Theorem 2 means $q > 1$. Hence

Corollary 16. *Let f be increasing and continuous. Assume moreover that $f(0) = 0$ and $\lim_{t\to\infty} f(t) = \infty$. Then for $1 < q \le 2$ there exists at least a nonnegative solution to*

$$\begin{cases} \Delta u + |\nabla u|^q = f(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega. \end{cases} \tag{7.6}$$

When $0 < q \leq 1$ and f verifies the Keller-Osserman condition (1.2) *there exists a nonnegative solution to* (7.6)*, and there exists none when*

$$\int_1^\infty \frac{ds}{f(s) + s^q} = \infty.$$

As before, the two important cases $f(t) = t^p$, $p > 0$ and $f(t) = e^t - 1$ give:

Corollary 17. *Let p, $q > 0$. Then the problem*

$$\begin{cases} \Delta u - |\nabla u|^q = u^q & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases}$$

has no nonnegative solutions if either $q > 2$ or if $p, q \leq 1$. When $1 < q \leq 2$ or $p > 1$, there exists a nonnegative solution. Moreover, the solution is unique if $1 \leq p < q \leq 2$.

Corollary 18. *Let $q > 0$. The problem*

$$\begin{cases} \Delta u - |\nabla u|^q = e^u - 1 & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases}$$

has no nonnegative solution if $q > 2$, while it has at least one when $0 < q \leq 2$.

Remark 5. As in the case considered in Section 7.1, the conditions in Corollary 16 are not exhaustive. For instance, the corollary cannot be applied to the function $f(t) = t(\log t)^\alpha$ when $1 < \alpha \leq 2$.

7.4. (P_-) with $g(t) = t^2 \log(t + 1)$. This is in our opinion one of the most interesting cases in our study, since it consists in an equation with a superquadratic growth in the gradient. It is not hard to see that condition (1.8) is verified. Moreover, also condition (1.10) in Theorem 2 holds, so that:

Corollary 19. *Let f be an increasing continuous function with $f(0) = 0$. Then the problem*

$$\begin{cases} \Delta u - |\nabla u|^2 \log(|\nabla u| + 1) = f(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases}$$

admits a nonnegative solution. If moreover $\frac{f(t)}{t}$ is increasing and

$$\lim_{t \to \infty} \frac{f(t)}{t^2 \log t} = 0,$$

then the solution is unique.

Appendix: Gradient bounds

It is important for the existence proofs in Section 4 to dispose of interior gradient bounds for solutions to the equations

$$\Delta u \pm g(|\nabla u|) = f(u) \quad \text{in } \Omega. \tag{A.1}$$

These bounds can be obtained independently of the growth of the function g. However, we need to impose restrictions (a) or (b) in the introduction, which are of technical nature.

By modifying the arguments in the Appendix to [35], where the case $g(t) = t^q$, $q > 0$ was considered, we are going to prove:

Theorem A. 1. *Let f and g be increasing continuous functions with $f(0) = g(0) = 0$. Assume g verifies one of the following two conditions:*

(a) There exists $t_0 > 0$ such that g is differentiable for $t \geq t_0$ and

$$\frac{g'(t)}{g(t)^2} \leq C\, t^{-\gamma}, \quad t \geq t_0$$

for some $\gamma > 2$, or

(b) There exists a positive constant C such that $|g(t)| \leq Ct^2$, for large t.

Let $u \in C^1(\Omega)$ be a nonnegative weak solution to (A.1)*. Then for every pair of smooth subdomains $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ there exists a constant C depending only on $|f(u)|_{L^\infty(\Omega'')}$, Ω', Ω'' and g such that*

$$\sup_{\Omega'} |\nabla u| \leq C. \tag{A.2}$$

Some interesting examples which fit into either of the previous conditions (a) or (b) are $g(t) = t^q$, for some $q > 0$, $g(t) = e^t - 1$ or $g(t) = t^q \log(t+1)$ for $q > 0$.

Proof of Theorem 1. As in the proof of Lemma 4, since it is easier to deal with $|\nabla u|^2$ than with $|\nabla u|$, we set $g(t) = \tilde{g}(t^2)$.

Let us begin with case (b), which is more or less classical. If $u \in C^1(\Omega)$ is a weak solution to (A.1), we have $\Delta u \in L^\infty_{\rm loc}(\Omega)$, so that by standard regularity $u \in W^{2,p}_{\rm loc}(\Omega)$ for every $p > 1$. We may use Theorem 6.5 in Chapter IV of [33] to obtain that for $\Omega' \subset\subset \Omega$, there exists $\alpha \in (0,1)$ and a constant $M > 0$ which only depends on $|u|_{L^\infty(\Omega')}$, $|f(u)|_{L^\infty(\Omega')}$, the constant C in (b) and the distance from Ω' to $\partial\Omega$, such that

$$|u|_{C^{1,\alpha}(\overline{\Omega}')} \leq M.$$

This shows (A.2) in case (b).

As for case (a), notice that we may assume $\lim_{t\to\infty} g(t) = \infty$, otherwise (b) holds. The condition on g implies

$$\frac{\tilde{g}'(t)}{\tilde{g}(t)^2} \leq C\, t^{-\frac{1+\gamma}{2}} \tag{A.3}$$

for large t and some positive constant C. By direct integration it is also easy to see that this entails

$$\tilde{g}(t) \geq Ct^{\frac{\gamma-1}{2}} \tag{A.4}$$

for large t and some positive C.

By approximation, as in Lemma 4, we may assume that $\tilde{g}$ and f are C^1. More precisely, let $\{f_k\}$, $\{g_k\}$ be sequences of C^1 functions such that $f_k \to f$, $g_k \to \tilde{g}$ uniformly in compacts of $\mathbb{R}$. We may also assume that f_k is increasing and g_k verifies condition (a) with a uniform constant C (notice that g is C^1 for large t, so we could take $g_k(t) = \tilde{g}(t)$ for large t if we wish). Let $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ be smooth subdomains. The problem

$$\begin{cases} \Delta v \pm g_k(|\nabla v|) = f_k(v) & \text{in } \Omega', \\ v = u & \text{on } \partial\Omega', \end{cases}$$

has a unique solution $v_k \in C^2(\overline{\Omega}')$. The existence follows exactly as in Proposition 5 and the uniqueness is a consequence of the comparison principle in [22]. Assuming we prove

$$\sup_{\Omega'} |\nabla v_k| \leq C \tag{A.5}$$

for a constant C which does not depend on k, we can argue as in Lemma 4 to deduce that (passing to a subsequence) $v_k \to u$ in $C^1(\overline{\Omega'})$. Letting $k \to \infty$ in (A.5) we obtain (A.2) for u.

Thus we assume f and $\tilde{g}$ are C^1. Next take $\varphi \in C_0^\infty(\Omega'')$ such that $\varphi \equiv 1$ in Ω', $0 \le \varphi \le 1$ and $|\Delta\varphi| \le C\varphi^\theta$, $|\nabla\varphi|^2 \le C\varphi^{1+\theta}$, for some $\theta \in (0,1)$ to be chosen later.

Let $w = |\nabla u|^2$. We have seen in Lemma 4 that w verifies the equation

$$\Delta w = 2|D^2u|^2 + 2f'(u)w \mp 2\tilde{g}'(w)\nabla u \nabla w$$

in Ω, so that the function $z := \varphi w$ verifies

$$\Delta z = 2\varphi|D^2u|^2 + 2\varphi f'(u)w \mp 2\tilde{g}'(w)\varphi\nabla u\nabla w + 2\nabla w\nabla\varphi + w\Delta\varphi.$$

Now notice that $z = 0$ outside Ω''. Hence there exists $x_0 \in \Omega''$ such that z achieves its maximum at x_0. Since $\Delta z(x_0) \le 0$ and $\nabla z(x_0) = 0$, we have

$$\begin{aligned} 2\varphi|D^2u|^2 &\le \mp 2\tilde{g}'(w)\nabla u\nabla\varphi w + \frac{|\nabla\varphi|^2}{\varphi}w - w\Delta\varphi \\ &\le 2\tilde{g}'(w)|\nabla u||\nabla\varphi|w + \frac{|\nabla\varphi|^2}{\varphi}w + w|\Delta\varphi| \\ &\le C\tilde{g}'(w)\varphi^{\frac{1+\theta}{2}}w^{\frac{3}{2}} + C\varphi^\theta w, \end{aligned} \tag{A.6}$$

at the point x_0, where we have used $\tilde{g}' \ge 0$ and $f' \ge 0$. On the other hand, thanks to Cauchy-Schwarz inequality, we have

$$(\Delta u)^2 = \left(\sum_{i=1}^N \partial_{ii}u\right)^2 \le N\sum_{i=1}^N(\partial_{ii}u)^2 \le N|D^2u|^2,$$

so that (A.6) implies that

$$\frac{2}{N}\varphi(\Delta u)^2 \le C\tilde{g}'(w)\varphi^{\frac{1+\theta}{2}}w^{\frac{3}{2}} + C\varphi^\theta w$$

at the point x_0.

Moreover, using the equation (A.1),

$$(\Delta u)^2 = (\tilde{g}(w) \mp f(u))^2 \ge (\tilde{g}(w) - C)^2 \ge C\tilde{g}(w)^2 - C,$$

where C depends on $\tilde{g}$ and $|f(u)|_{L^\infty(\Omega'')}$, so that

$$\varphi\tilde{g}(w)^2 \le C\varphi + C\tilde{g}'(w)\varphi^{\frac{1+\theta}{2}}w^{\frac{3}{2}} + C\varphi^\theta w$$

at x_0. Using (A.3) and $w = z/\varphi$ we get

$$\tilde{g}(w)^2 \le C + C\tilde{g}(w)^2 z^{1-\frac{\gamma}{2}}\varphi^{\frac{1+\theta}{2}-2+\frac{\gamma}{2}} + C\varphi^{\theta-2}z. \tag{A.7}$$

We now choose $\theta \ge 3-\gamma$, which is always possible since $3-\gamma < 1$. Then (A.7) becomes

$$\tilde{g}(w)^2 \le C + C\tilde{g}(w)^2 z^{1-\frac{\gamma}{2}} + C\varphi^{\theta-2}z. \tag{A.8}$$

Now assume $Cz(x_0)^{1-\frac{\gamma}{2}} \le 1/2$, since on the contrary there is nothing to prove. Then (A.8) implies $\tilde{g}(w)^2 \le C + C\varphi^{\theta-2}z$ and using (A.4) we arrive at $w^{\gamma-1} \le C + C\varphi^{\theta-2}z$, that is

$$z^{\gamma-1} \le C\varphi^{\gamma-1} + C\varphi^{\theta+\gamma-3}z \le C + Cz.$$

Taking into account that $\gamma > 2$, we obtain an upper bound for $z(x_0)$ which only depends on $|f(u)|_{L^\infty(\Omega'')}$, Ω', Ω'' and g. Since x_0 is a point where z achieves its maximum

$$\varphi|\nabla u|^2 \le C \quad \text{in } \Omega,$$

and using that $\varphi \equiv 1$ in Ω', we arrive at (A.2). This concludes the proof. □

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