EXISTENCE AND UNIQUENESS OF SOLUTIONS TO NONLINEAR ELLIPTIC EQUATIONS WITHOUT GROWTH CONDITIONS AT INFINITY

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Abstract. In this paper we consider the nonlinear elliptic problem $-\Delta u + |u|^{p-1}u + |\nabla u|^q = f$ in $\mathbb{R}^N$, where $p > 1$ and $q > 0$. We show that if the function f belongs to $L^r_{\text{loc}}(\mathbb{R}^N)$ for some suitable $r \geq 1$, then there exists a distributional solution to the equation, independently of the behavior of f at infinity. We also analyze the uniqueness of this solution in some cases.

1. Introduction

In [6], the following somewhat surprising result was obtained: if $p > 1$ and $f \in L^1_{\text{loc}}(\mathbb{R}^N)$, then there exists a unique distributional solution to

$$-\Delta u + |u|^{p-1}u = f \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

The surprising fact is that the existence or uniqueness of solutions to (1.1) do not depend on the behavior of f at infinity, but strongly relies on the fact that $p > 1$.

The question of existence and uniqueness of solutions without prescribing a growth on f at infinity has been subsequently considered for more general equations than (1.1), obtained when the Laplacian is substituted by a divergence operator of $p-$Laplacian type (see [4] or [12]) or by a fully nonlinear operator (cf. [7]). The extension to parabolic equations has been also studied in [5] and [13].

Our intention in the present paper is to analyze whether the existence and uniqueness features in (1.1) still hold when we introduce a term that depends on the gradient in the equation. More precisely, we will be interested in the problem

$$-\Delta u + |u|^{p-1}u + |\nabla u|^q = f \quad \text{in } \mathbb{R}^N \tag{1.2}$$

where $p > 1$, $q > 0$ and $f \in L^1_{\text{loc}}(\mathbb{R}^N)$. Although it will be clear from our proofs below that some slightly more general problems can be considered, we will restrict our attention to (1.2) for simplicity.

By a solution to (1.2) we understand a function $u \in L^p_{\text{loc}}(\mathbb{R}^N)$ such that $|\nabla u|^q \in L^1_{\text{loc}}(\mathbb{R}^N)$, and verifies (1.2) in the distributional sense, i. e.

$$-\int u\Delta\phi + \int (|u|^{p-1}u + |\nabla u|^q)\phi = \int f\phi$$

for every $\phi \in C_0^\infty(\mathbb{R}^N)$. It is then known that since $\Delta u \in L^1_{\text{loc}}(\mathbb{R}^N)$ we have $u \in W^{1,s}_{\text{loc}}(\mathbb{R}^N)$ for every s such that $1 \leq s < N/(N-1)$. As it is usual, a slight increase of regularity in f will reflect in a gain of regularity for u.

Let us state our results. We begin with the case where $0 < q < 2p/(p+1)$, where the regularity $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ is enough to obtain a solution. In this case, the gradient term does not "interfere too much" with the structure of (1.1).

Theorem 1. *Let $p > 1$ and $0 < q < \frac{2p}{p+1}$. Then for every $f \in L^1_{\rm loc}(\mathbb{R}^N)$, there exists a distributional solution u to the problem*

$$-\Delta u + |u|^{p-1}u + |\nabla u|^q = f \quad in\ \mathbb{R}^N.$$

This solution verifies $u \in L^p_{\rm loc}(\mathbb{R}^N) \cap W^{1,s}_{\rm loc}(\mathbb{R}^N)$ for every $s \in \big(1, \max\{\frac{2p}{p+1}, \frac{N}{N-1}\}\big)$. Moreover, if $f \geq 0$ we have $u \geq 0$.

When $q \geq 2p/(p+1)$, the $L^1_{\rm loc}$ regularity for f seems not enough to ensure existence. Also, let us remark that the sign of f is an important issue here (mainly when $q > 2$). This is due to the fact that the equation in (1.2) is not invariant under the change of u by $-u$, as happens with (1.1). Thus we are restricting ourselves in the present paper to the case where $f \geq 0$, delaying the study of a negative f to a future work. We remark that the differences between a positive and a negative f can be seen even when the problem is posed in a bounded domain in $\mathbb{R}^N$. Our next result is indeed valid in the whole range $q > 1$.

Theorem 2. *Let $p, q > 1$ and $f \in L^r_{\rm loc}(\mathbb{R}^N)$ for some $r > N$ with $f \geq 0$. Then there exists a strong solution $u \in C^1(\mathbb{R}^N) \cap W^{2,r}_{\rm loc}(\mathbb{R}^N)$ to the equation*

$$-\Delta u + |u|^{p-1}u + |\nabla u|^q = f \quad in\ \mathbb{R}^N$$

which is in addition positive.

Next, we analyze how the condition $f \in L^r_{\rm loc}(\mathbb{R}^N)$ with $r > N$ can be weakened. For this purpose, we consider a radially symmetric, nonnegative function f and try to construct radially symmetric, nonnegative solutions. It turns out that in this framework it is enough to have $r > 1$, provided that $1 < q < N/(N-1)$. Let us mention in passing that next theorem is valid in the full range $p > 1$, but it only gives better results than Theorem 1 when $p < N/(N-2)$, $q \geq 2p/(p+1)$.

Theorem 3. *Let $1 < p < \frac{N}{N-2}$ and $\frac{2p}{p+1} \leq q < \frac{N}{N-1}$. For every radially symmetric, nonnegative function $f \in L^r_{\rm loc}(\mathbb{R}^N)$, $r > 1$, there exists a radially symmetric, nonnegative distributional solution u to the equation*

$$-\Delta u + u^p + |\nabla u|^q = f \quad in\ \mathbb{R}^N.$$

Let us quote that the proof of existence of solutions in all theorems is achieved by first considering the problem in a smooth bounded domain of $\mathbb{R}^N$, complemented with a Dirichlet boundary condition. The essential step is then to obtain good estimates for the solutions and their gradients.

Finally, we analyze the question of uniqueness of the solutions constructed before. Due to the lack of regularity of such solutions in the case $0 < q \leq 1$, it is difficult to establish their uniqueness. We will be able to do it only if the regularity of f is slightly improved.

Theorem 4. *Assume $f \in L^r_{\rm loc}(\mathbb{R}^N)$ for some $r > N$ and is nonnegative. Then if $0 < q < p$, problem (1.2) admits a unique solution in $W^{1,\infty}_{\rm loc}(\mathbb{R}^N)$, which is in addition nonnegative. This solution is indeed in $C^1(\mathbb{R}^N) \cap W^{2,r}_{\rm loc}(\mathbb{R}^N)$.*

It is worthy of mention that the condition $q < p$ is optimal in the uniqueness assertion, since when $q \geq p$ infinitely many smooth solutions can be constructed when $f = 0$ (see Remark 4).

For the proof of Theorem 4, we follow a device used in [6], with a significant variation: we use as a tool the minimal solution U_R to the boundary blow-up problem

$$\begin{cases} -\Delta U + c\,U^p - d|\nabla U|^q = 0 & \text{in } B_R \\ U = \infty & \text{on } \partial B_R, \end{cases}$$

where $c, d > 0$, which was shown to exist in [1]. A worth noticing property is that $U_R \to 0$ uniformly in compacts of $\mathbb{R}^N$ as $R \to \infty$.

The rest of the paper is organized as follows: in Section 2 we consider problem (1.2) in smooth bounded domains of $\mathbb{R}^N$ with a Dirichlet boundary condition, while Section 3 is dedicated to the obtention of local estimates for these approximate solutions and their gradients. Finally the proofs of Theorems 1, 2, 3 and 4 are performed in Section 4.

2. A problem in bounded domains

As we have already mentioned in the Introduction, the construction of solutions to (1.2) relies in the solvability of a related Dirichlet problem in a smooth bounded domain Ω of $\mathbb{R}^N$. The purpose of this brief section is to analyze such problem. Thus we will consider

$$\begin{cases} -\Delta u + |u|^{p-1}u + |\nabla u|^q = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $p > 1$ and $q > 0$. In the present context, we may always take the function f to be smooth enough, hence we will assume $f \in C^\infty(\overline{\Omega})$. The result we will need is the following:

Theorem 5. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and $f \in C^\infty(\overline{\Omega})$, $p > 1$, $q > 0$. When $q > 2$, assume additionally that $f \geq 0$. Then problem* (2.1) *admits a unique classical solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Moreover, when $f \geq 0$, we also have $u > 0$ in Ω.*

Proof. It is clear that the unique solution $\bar{u}$ to the problem

$$\begin{cases} -\Delta u + |u|^{p-1}u = |f|_\infty & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is a supersolution to (2.1), which, according to the strong maximum principle, verifies $\bar{u} > 0$ in Ω. Next, assume $0 < q \leq 2$. We claim that the problem

$$\begin{cases} -\Delta u + |u|^{p-1}u + |\nabla u|^q = -|f|_\infty & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

has a unique (negative) solution. Indeed, setting $\theta = |f|_\infty^{\frac{1}{p}}$ and $v = u + \theta$, we can see that (2.2) is equivalent to

$$\begin{cases} \Delta v - |\nabla v|^q = h(v) & \text{in } \Omega \\ v = \theta & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where $h(v) = \theta^p + |v-\theta|^{p-1}(v-\theta)$ is increasing and verifies $h(0) = 0$. From Proposition 5 in [1], we deduce that problem (2.3) has unique solution $\underline{v}$, which verifies $\underline{v} < \theta$ in Ω from the strong maximum principle. Letting $\underline{u} = \underline{v} - \theta$ we obtain the unique solution to (2.2), which is a subsolution to (2.1).

Since $0 < q \leq 2$, we obtain by standard results (see for instance [2] or [16]) the existence of a weak solution $u \in C^1(\overline{\Omega})$ to (2.1) which verifies $\underline{u} < u < \overline{u}$ in Ω. By classical regularity, we also have $u \in C^2(\Omega)$, although this regularity can be improved depending on the values of p and q.

When $q > 2$ and $f \geq 0$, we may still take $\bar{u}$ as a supersolution and $\underline{u} = 0$ as a subsolution, and we obtain the existence of a solution u verifying $0 < u < \overline{u}$ in Ω thanks to Theorem III.1 in [14]. □

3. Interior estimates for solutions and their gradients

In order to construct solutions to (1.2), we need local bounds for solutions and their gradients. The obtention of these bounds can be achieved when $q < 2p/(p+1)$ with the mere assumption $f \in L^1_{\rm loc}(\mathbb{R}^N)$, but the case $q \geq 2p/(p+1)$ is not so straightforward and a different strategy must be employed.

We begin by considering the bounds for weak solutions when $f \in L^1_{\rm loc}(\mathbb{R}^N)$ and $0 < q < 2p/(p+1)$. Recall that $u \in H^1(B_{2R})$ is a weak solution to $-\Delta u + |u|^{p-1}u + |\nabla u|^q = f$ in B_{2R} if

$$\int \nabla u \nabla \phi + (|u|^{p-1}u + |\nabla u|^q)\phi = \int f\phi$$

for every $\phi \in C_0^\infty(B_{2R})$. The proof of the next result is inspired in [4].

Theorem 6. *Let $p > 1$ and $0 < q < \frac{2p}{p+1}$. Then for every $R > 0$ there exists a constant $C = C(R) > 0$ such that for every weak solution $u \in H^1(B_{2R})$ to*

$$-\Delta u + |u|^{p-1}u + |\nabla u|^q = f \quad \text{in } B_{2R} \tag{3.1}$$

with $f \in L^1_{\rm loc}(\mathbb{R}^N)$ we have

$$\int_{B_R} |u|^p \leq C\left(\int_{B_{2R}} |f| + 1\right). \tag{3.2}$$

Also, for every $s \in (0, \frac{2p}{p+1})$ there exists $C = C(s,R)$ such that

$$\int_{B_R} |\nabla u|^s \leq C\left(\int_{B_{2R}} |f| + 1\right). \tag{3.3}$$

Proof. For $m > 0$ we introduce the function

$$\phi_m(\sigma) = m\int_0^\sigma \frac{dt}{(1+|t|)^{m+1}}, \qquad \sigma \in \mathbb{R},$$

which is odd and verifies $\left|\phi_m(\sigma)\right| \leq 1$. Choose $\theta \in C_0^\infty(B_{2R})$ verifying $0 \leq \theta \leq 1$ and $\theta \equiv 1$ in B_R. We take $\phi_m(u)\theta^\alpha$ as a test function in the weak formulation of (3.1), where $\alpha > 0$ is to be selected later on, to obtain

$$\begin{aligned} m\int \frac{|\nabla u|^2}{(1+|u|)^{m+1}}\theta^\alpha &+ \int |u|^{p-1}u\phi_m(u)\theta^\alpha \\ &\leq \int |f|\theta^\alpha + C\int \theta^{\alpha-1}|\nabla u| + \int \theta^\alpha |\nabla u|^q, \end{aligned}$$

thanks to the definition of ϕ_m. We adopt the usual convention that the letter C denotes different constants, not depending on u nor f. Observe now that from Young's inequality we have

$$\theta^{\alpha-1}|\nabla u| \leq \varepsilon\frac{|\nabla u|^2\theta^\alpha}{(1+|u|)^{m+1}} + C(\varepsilon)\,\theta^{\alpha-2}(1+|u|)^{m+1}$$

for every $\varepsilon > 0$, where $C(\varepsilon)$ depends only on ε. On the other hand, we can take q_0 such that $\max\{1, q\} < q_0 < \frac{2p}{p+1}$, and since $|\nabla u|^q \le 1 + |\nabla u|^{q_0}$, we obtain again by Young's inequality

$$|\nabla u|^q \le 1 + \varepsilon \frac{|\nabla u|^2}{(1+|u|)^{m+1}} + C(\varepsilon)(1+|u|)^{\frac{(m+1)q_0}{2-q_0}}.$$

Hence, if we fix $\varepsilon > 0$ sufficiently small, we arrive at

$$\begin{aligned}
&\int \frac{|\nabla u|^2}{(1+|u|)^{m+1}}\theta^\alpha + \int |u|^{p-1}u\phi_m(u)\theta^\alpha \\
&\le C\int_{B_{2R}} |f| + C\int (1+|u|)^{m+1}\theta^{\alpha-2} + C\int (1+|u|)^{\frac{(m+1)q_0}{2-q_0}}\theta^\alpha + C \\
&\le C\int_{B_{2R}} |f| + C\int (1+|u|)^{\frac{(m+1)q_0}{2-q_0}}\theta^{\alpha-2} + C,
\end{aligned}$$

since $q_0/(2-q_0) > 1$. On the other hand, it is easily seen that $|u|^{p-1}u\phi_m(u) \ge C|u|^p - 1$ for $u \in \mathbb{R}$. Hence

$$\int \frac{|\nabla u|^2}{(1+|u|)^{m+1}}\theta^\alpha + \int |u|^p\theta^\alpha \le C\int_{B_{2R}} |f| + C\int (1+|u|)^{\frac{(m+1)q_0}{2-q_0}}\theta^{\alpha-2} + C.$$

A further application of Young's inequality gives

$$\int (1+|u|)^{\frac{(m+1)q_0}{2-q_0}}\theta^{\alpha-2} \le \varepsilon\int (1+|u|)^p\theta^\alpha + C(\varepsilon)\int \theta^{\alpha-\frac{2p}{p-\mu}},$$

where $\mu = \dfrac{(m+1)q_0}{2-q_0}$. We note that we can achieve $\mu < p$ if we choose m small enough. Taking $\alpha \ge \dfrac{2p}{p-\mu}$ and recalling that $\theta \equiv 1$ in B_R, we obtain

$$\int_{B_R} \frac{|\nabla u|^2}{(1+|u|)^{m+1}} + \int_{B_R} |u|^p \le C\left(\int_{B_{2R}} |f| + 1\right). \tag{3.4}$$

Finally observe that (3.4) holds for all $m > 0$ since it holds for small m and the left-hand side is decreasing in m.

Now (3.2) follows immediately from (3.4). With regard to (3.3), we can use Hölder's inequality for every $s \in (0, \frac{2p}{p+1})$, to have

$$\begin{aligned}
\int_{B_R} |\nabla u|^s &= \int_{B_R} \frac{|\nabla u|^s}{(1+|u|)^\nu}(1+|u|)^\nu \\
&\le \left(\int_{B_R} \frac{|\nabla u|^2}{(1+|u|)^{\frac{2\nu}{s}}}\right)^{\frac{s}{2}} \left(\int_{B_R} (1+|u|)^{\frac{2\nu}{2-s}}\right)^{\frac{2-s}{2}},
\end{aligned} \tag{3.5}$$

for every $\nu > 0$. Notice that we can select ν so that $\frac{2\nu}{s} > 1$, $\frac{2\nu}{2-s} \le p$, since this is equivalent to

$$\frac{s}{2} < \nu \le \frac{p(2-s)}{2}.$$

This election is possible since $s < \frac{2p}{p+1}$. Thus (3.3) follows at once from (3.5) and (3.4). The proof is concluded. □

Remark 1. When $f \in L^r_{\rm loc}(\mathbb{R}^N)$ for some $r > 1$, we can obtain better estimates for weak nonnegative, bounded solutions (these last two assumptions are only made for the sake of simplicity in the present proof; they do not seem necessary). Indeed, if u is such a solution we have

$$\int_{B_R} u^{pr} \le C\left(\int_{B_{2R}} |f|^r + 1\right). \tag{3.6}$$

To see this, we take as a test function in (3.1) $(u+\varepsilon)^m\theta^\alpha$, where θ is as in the previous proof, $m = p/(r'-1)$, $\varepsilon > 0$ is small and $\alpha > 0$ is to be chosen. This leads to

$$m\int \theta^\alpha (u+\varepsilon)^{m-1}|\nabla u|^2 + \int \theta^\alpha u^p (u+\varepsilon)^m \le \int |f|\theta^\alpha (u+\varepsilon)^m + \alpha\int \theta^{\alpha-1}(u+\varepsilon)^m|\nabla u||\nabla\theta|.$$

From Young's inequality applied to the last integral we obtain

$$\alpha\int \theta^{\alpha-1}(u+\varepsilon)^m|\nabla u||\nabla\theta| \le \frac{m}{2}\int \theta^\alpha(u+\varepsilon)^{m-1}|\nabla u|^2 + C\int \theta^{\alpha-2}(u+\varepsilon)^{m+1},$$

where C is a positive constant. Hence

$$\int \theta^\alpha u^p(u+\varepsilon)^m \le \int |f|\theta^\alpha(u+\varepsilon)^m + C\int \theta^{\alpha-2}(u+\varepsilon)^{m+1}.$$

We can let first $\varepsilon \to 0$ and then apply Young's inequality to have $|f|u^m \le \frac{1}{2}u^{pr} + C|f|^r$, so that

$$\frac{1}{2}\int \theta^\alpha u^{pr} \le C\int |f|^r\theta^\alpha + C\int \theta^{\alpha-2}u^{m+1}. \tag{3.7}$$

Next observe that if we employ once again Young's inequality in the last integrand we have $\theta^{\alpha-2}u^{m+1} \le \frac{1}{4}\theta^\alpha u^{pr} + C\theta^{\alpha-2p'r}$, and hence (3.7) gives, choosing $\alpha > 2p'r$,

$$\int \theta^\alpha u^{pr} \le C\left(\int |f|^r\theta^\alpha + 1\right).$$

We obtain (3.6) since $0 \le \theta \le 1$ and $\theta = 1$ in B_R.

In the complementary case where $q \ge 2p/(p+1)$, we will impose an extra amount of regularity on f. Namely, we will assume that $f \in L^r_{\rm loc}(\mathbb{R}^N)$ for some $r > N$. Although the estimates for the solutions can be achieved arguing as in [6], it is not so clear how to obtain appropriate bounds for the gradients to pass to the limit. Hence our approach will be completely different than in the previous case: it will be a mixture of those in [11] and [15], and is indeed valid in the whole range $q > 1$.

Theorem 7. *Let $p, q > 1$, $f \in C^1(\mathbb{R}^N)$ and fix $r > N$. If $u \in C^3(B_{4R})$ is a nonnegative classical solution to*

$$-\Delta u + u^p + |\nabla u|^q = f \quad \text{in } B_{4R}$$

then there exists a positive constant depending on R and $|f|_{L^r(B_{4R})}$ such that

$$\sup_{B_R}(u + |\nabla u|) \le C. \tag{3.8}$$

Proof. We first claim that there exists a positive constant C not depending on f nor on u such that

$$\int_{B_{2R}} u^p \le C\left(\int_{B_{4R}} |f| + 1\right). \tag{3.9}$$

To prove (3.9) we can argue exactly as in [6]. Notice that $-\Delta u + u^p \le f$ in $\mathbb{R}^N$. Now take a cut-off function $\xi \in C^\infty(B_{4R})$ verifying $0 \le \xi \le 1$ and $\xi \equiv 1$ in B_{2R}. Testing the equation with ξ^α, where $\alpha > 0$ will be chosen later on, we have

$$\begin{aligned}\int u^p \xi^\alpha &\le \int_{B_{4R}} |f| + \int u\Delta\xi^\alpha \le \int_{B_{4R}} |f| + C\int u\xi^{\alpha-2}\\ &\le \int_{B_{4R}} |f| + C\Big(\int u^p\, \xi^{(\alpha-2)p}\Big)^{\frac{1}{p}},\end{aligned}$$

where C is a positive constant (depending only on α and ξ) and Hölder's inequality has been used. Setting $\alpha = \frac{2p}{p-1}$, we get

$$\int u^p\xi^\alpha \le \int_{B_{2R}} |f| + C\Big(\int u^p\xi^\alpha\Big)^{\frac{1}{p}}$$

so that (3.9) follows by recalling that $\xi \equiv 1$ in B_{2R}. Now observe that $-\Delta u \le f$ in B_{2R}, so that we may apply Theorem 8.17 in [9] to arrive at $u \in L^\infty(B_R)$, with the bound

$$\sup_{B_R} u \le C$$

where C depends on R, $|f|_{L^r(B_{4R})}$ and on p.

Our next concern will be to obtain estimates for the gradient of u. Let $w = |\nabla u|^2$, and observe that w is a smooth function. It is not hard to see that

$$-\Delta w + q|\nabla u|^{q-2}\nabla u\nabla w = -2|D^2u|^2 + 2\nabla u\nabla f - 2pu^{p-1}w$$

in $\mathbb{R}^N$. Next take a smooth cut-off function φ such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ in B_R, $\varphi \equiv 0$ in $\mathbb{R}^N \backslash B_{2R}$, and $|\Delta\varphi| \le C\varphi^\theta$, $|\nabla\varphi|^2 \le C\varphi^{1+\theta}$, for some positive constant C and a certain $\theta \in (0,1)$ to be chosen later. We have

$$\begin{aligned}&-\Delta(\varphi w) + qw^{\frac{q-2}{2}}\nabla u\nabla(\varphi w) + 2\frac{\nabla\varphi}{\varphi}\nabla(\varphi w) + 2|D^2u|^2\varphi\\ &= qw^{\frac{q-2}{2}}\nabla u\nabla\varphi w - \Delta\varphi w + 2\nabla u\nabla f\varphi - 2pu^{p-1}w + 2\frac{|\nabla\varphi|^2}{\varphi}w\end{aligned}$$

in $\mathbb{R}^N$. Taking now $m > 0$, and using $(\varphi w)^m$ as a test function we obtain

$$\begin{aligned}&m\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} + q\int w^{\frac{q-2}{2}}\nabla u\nabla(\varphi w)(\varphi w)^m\\ &+2\int \frac{\nabla\varphi}{\varphi}\nabla(\varphi w)(\varphi w)^m + 2\int |D^2u|^2\varphi(\varphi w)^m\\ &= q\int w^{\frac{q-2}{2}}\nabla u\nabla\varphi(\varphi w)^m w - \int \Delta\varphi w(\varphi w)^m\\ &+2\int \nabla u\nabla f\varphi(\varphi w)^m - 2p\int u^{p-1}w(\varphi w)^m + 2\int \frac{|\nabla\varphi|^2}{\varphi}w(\varphi w)^m.\end{aligned}$$

We next notice that, thanks to Cauchy-Schwarz inequality:

$$\begin{aligned}|D^2u|^2 &\ge \frac{1}{N}(\Delta u)^2 \ge \frac{1}{N}(|\nabla u|^q + u^p - f)^2\\ &\ge \frac{1}{2N}\big(|\nabla u|^q + u^p\big)^2 - \frac{2}{N}|f|^2 \ge \frac{1}{2N}|\nabla u|^{2q} - \frac{2}{N}|f|^2.\end{aligned}$$

It follows that

$$\begin{aligned} & m\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} + q\int w^{\frac{q-2}{2}}\nabla u\nabla(\varphi w)(\varphi w)^m \\ & \quad +2\int \frac{\nabla\varphi}{\varphi}\nabla(\varphi w)(\varphi w)^m + \frac{1}{2N}\int w^q\varphi(\varphi w)^m + \int |D^2u|^2\varphi(\varphi w)^m \\ & \le q\int w^{\frac{q-2}{2}}|\nabla\varphi| w(\varphi w)^m \quad + \int |\Delta\varphi| w(\varphi w)^m + \frac{4}{N}\int |f|^2(\varphi w)^m \\ & \quad +2\int \frac{|\nabla\varphi|^2}{\varphi}w(\varphi w)^m + 2\int \nabla u\nabla f\varphi(\varphi w)^m + C, \end{aligned}$$

where for simplicity from now on, the letter C represents a generic constant independent of u, f, m. Now by the choice we have made of φ:

$$\begin{aligned} q\int w^{\frac{q-1}{2}}|\nabla\varphi| w(\varphi w)^m &\le C\int \varphi^{m+\frac{\theta+1}{2}}w^{m+\frac{q+1}{2}} \le C\int \varphi^{m+\theta}w^{m+\frac{q+1}{2}} \\ \int |\Delta\varphi| w(\varphi w)^m &\le C\int \varphi^{m+\theta}w^{m+1} \\ 2\int \frac{|\nabla\varphi|^2}{\varphi}w(\varphi w)^m &\le C\int \varphi^{m+\theta}w^{m+1}, \end{aligned}$$

hence

(3.10)

$$\begin{aligned} & m\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} + q\int w^{\frac{q-2}{2}}\nabla u\nabla(\varphi w)(\varphi w)^m \\ & \quad +2\int \frac{\nabla\varphi}{\varphi}\nabla(\varphi w)(\varphi w)^m + \frac{1}{2N}\int \varphi^{m+1}w^{m+q} + \int |D^2u|^2\varphi(\varphi w)^m \\ & \le C\int \varphi^{m+\theta}w^{m+\frac{q+1}{2}} + C\int \varphi^{m+\theta}w^{m+1} + \frac{4}{N}\int |f|^2(\varphi w)^m + 2\int \nabla u\nabla f\varphi(\varphi w)^m + C \\ & \le C\int \varphi^{m+\theta}w^{m+\frac{q+1}{2}} + \frac{4}{N}\int |f|^2(\varphi w)^m + 2\int \nabla u\nabla f\varphi(\varphi w)^m + C. \end{aligned}$$

The next step is to estimate the integrals in the right-hand side of (3.10) with the aid of Young's inequality in the form $ab \le \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$ for a, $b > 0$ and arbitrary $\varepsilon > 0$. With regard to the integrals that do not contain f:

$$\begin{aligned} -q\int w^{\frac{q-2}{2}}\nabla u\nabla(\varphi w)(\varphi w)^m &\le q\int w^{\frac{q-1}{2}}|\nabla(\varphi w)|(\varphi w)^m \\ &\le \varepsilon\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} + \frac{q}{4\varepsilon}\int w^{q-1}(\varphi w)^{m+1} \\ &= \varepsilon\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} + \frac{q}{4\varepsilon}\int \varphi^{m+1}w^{m+q} \end{aligned}$$

and

$$
\begin{aligned}
-2\int \frac{\nabla\varphi}{\varphi}\nabla(\varphi w)(\varphi w)^m &\le 2\int \frac{|\nabla\varphi|}{\varphi}|\nabla(\varphi w)|(\varphi w)^m \\
&\le C\int \varphi^{\frac{\theta-1}{2}}|\nabla(\varphi w)|(\varphi w)^m \\
&\le \varepsilon\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} + \frac{C}{\varepsilon}\int \varphi^{\theta-1}(\varphi w)^{m+1} \\
&\le \varepsilon\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} + \frac{C}{\varepsilon}\int \varphi^{m+\theta}w^{m+1}.
\end{aligned}
$$

Next we consider the integral involving f in the right-hand side of (3.10). Integrating by parts and using Cauchy-Schwarz inequality we have

$$
\begin{aligned}
2\int \nabla u\nabla f\varphi(\varphi w)^m &= -2\int f\mathrm{div}\big(\nabla u\varphi(\varphi w)^m\big) \\
&= -2\int f\Delta u\varphi(\varphi w)^m - 2\int f\nabla u\nabla\varphi(\varphi w)^m - 2\int f\nabla u\varphi\nabla(\varphi w)^m \\
&\le 2\int |f||\Delta u|\varphi(\varphi w)^m + 2\int |f||\nabla u||\nabla\varphi|(\varphi w)^m \\
&\quad +2m\int |f||\nabla u|\varphi(\varphi w)^{m-1}|\nabla(\varphi w)| \\
&\le \int |D^2u|^2\varphi(\varphi w)^m + C\int |f|^2\varphi(\varphi w)^m + \int |f|^2(\varphi w)^m \\
&\quad + \int |\nabla u|^2|\nabla\varphi|^2(\varphi w)^m + \frac{m}{2}\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} \\
&\quad +Cm\int |f|^2\varphi|\nabla u|^2(\varphi w)^{m-1} \\
&\le \int |D^2u|^2\varphi(\varphi w)^m + Cm\int |f|^2(\varphi w)^m + \frac{m}{2}\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} \\
&\quad +C\int \varphi^{m+1+\theta}w^{m+1}.
\end{aligned}
$$

Hence, plugging everything into (3.10), we get

$$
\begin{aligned}
&\left(\frac{m}{2} - 2\varepsilon\right)\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} + \left(\frac{1}{2N} - \frac{C}{\varepsilon}\right)\int \varphi^{m+1}w^{m+q} \\
&\qquad \le C\int \varphi^{m+\theta}w^{m+1} + C\int \varphi^{m+\theta}w^{m+\frac{q+1}{2}} + Cm\int |f|^2(\varphi w)^m + C.
\end{aligned}
$$

Choosing and fixing a small enough ε and then a large m it follows that

$$
\begin{aligned}
&\frac{m}{3}\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} + \frac{1}{4N}\int \varphi^{m+1}w^{m+q} \\
&\qquad \le C\int \varphi^{m+\theta}w^{m+\frac{q+1}{2}} + C\int |f|^2(\varphi w)^m + C.
\end{aligned}
\tag{3.11}
$$

Now, we observe that it is possible to choose $\theta \in (0,1)$ independent on m such that

$$
\frac{m+\theta}{m+\frac{q+1}{2}} > \frac{m+1}{m+q}.
$$

With this choice of θ we obtain $\varphi^{m+\theta}w^{m+\frac{q+1}{2}} \leq \frac{1}{4N}\varphi^{m+1}w^{m+q}+C$, and hence from (3.11)

$$\frac{m}{3}\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} \leq C\int |f|^2(\varphi w)^m + C. \tag{3.12}$$

On the other hand, by means of Sobolev's inequality:

$$\frac{m}{3}\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} = \frac{4m}{3(m+1)^2}\int |\nabla(\varphi w)^{\frac{m+1}{2}}|^2 \geq C\left(\int (\varphi w)^{\frac{(m+1)N}{N-2}}\right)^{\frac{N-2}{N}},$$

where the constant C depends also on m. Applying Hölder's inequality to the integral containing f in (3.12):

$$\int |f|^2(\varphi w)^m \leq \left(\int (\varphi w)^{\frac{(m+1)N}{N-2}}\right)^{\frac{m}{m+1}\frac{N-2}{N}}\left(\int_{B_{2R}} |f|^{2\beta}\right)^{\frac{1}{\beta}} \tag{3.13}$$

where $\beta = \left(\dfrac{m+1}{m}\dfrac{N}{N-2}\right)'$. Since $\beta \to N/2 < r/2$ as $m \to \infty$, we may choose and fix m so large that $2\beta < r$, so that the last integral in (3.13) is controlled by $|f|_{L^r(B_{2R})}$. Hence from (3.12):

$$\left(\int (\varphi w)^{\frac{(m+1)N}{N-2}}\right)^{\frac{N-2}{N}} \leq C\left(\int (\varphi w)^{\frac{(m+1)N}{N-2}}\right)^{\frac{N-2}{N}\frac{m}{m+1}} + C$$

which immediately implies

$$\int (\varphi w)^{\frac{(m+1)N}{N-2}} \leq C.$$

Taking into account that $\varphi \equiv 1$ in B_R and the definition $w = |\nabla u|^2$ we finally have

$$\int_{B_R} |\nabla u|^{\frac{2(m+1)N}{N-2}} \leq C.$$

Since m can be taken to be arbitrarily large, we obtain local bounds for $|\nabla u|$ in L^s for every $s > 1$. Finally, the equation implies that $-\Delta u = h$ where $h \in L^r_{\text{loc}}(\mathbb{R}^N)$ for some $r > N$. We have then (3.8) thanks to Theorem 5.2 in [10]. □

Remark 2. Estimates for $|\nabla u|$ in L^q can be obtained as in [6] when $q > 1$. We take a cut-off function $\xi \in C^\infty(B_{2R})$ with $0 \leq \xi \leq 1$, $\xi \equiv 1$ in B_R. For $\alpha > 0$ to be chosen we test the equation with ξ^α and since u is nonnegative:

$$\begin{aligned}\int |\nabla u|^q\xi^\alpha &\leq \int_{B_{2R}} |f| - \alpha\int \xi^{\alpha-1}\nabla u\nabla\xi \\ &\leq \int_{B_{2R}} |f| + C\int \xi^{\alpha-1}|\nabla u| \\ &\leq \int_{B_{2R}} |f| + C\left(\int \xi^{(\alpha-1)q}|\nabla u|^q\right)^{\frac{1}{q}},\end{aligned}$$

thanks to Hölder's inequality. Taking $\alpha = (\alpha-1)q$, i.e. $\alpha = \frac{q}{q-1}$, we obtain

$$\int |\nabla u|^q\xi^\alpha \leq \int_{B_{2R}} |f| + C\left(\int \xi^\alpha|\nabla u|^q\right)^{\frac{1}{q}},$$

where C depends on R, and in particular this gives bounds for $|\nabla u|^q$ in $L^1(B_R)$. However, these estimates are not enough to pass to the limit in the proof of Theorem 2.

Finally, we consider a particular case where appropriate estimates for the gradient of the solutions can be obtained with only a slight increase in regularity on f. We will restrict ourselves to a radially symmetric situation.

Theorem 8. *Assume $p > 1$ and $1 < q < \frac{N}{N-1}$. Let $f \in L^r_{\mathrm{loc}}(\mathbb{R}^N)$ for some $r > 1$ be radially symmetric. Then for every $R > 0$ there exists $\delta > 0$ and a positive constant $C = C(\delta, R, |f|_{L^r(B_{2R})})$ such that for every radially symmetric smooth nonnegative solution to*

$$-\Delta u + u^p + |\nabla u|^q = f \quad \text{in } B_{2R}$$

we have

$$\int_{B_R} |\nabla u|^{q+\delta} \leq C. \tag{3.14}$$

Proof. Since u is radially symmetric and nonnegative, we have

$$-u'' - \frac{N-1}{s}u' + |u'|^q \leq |f|,$$

where $s = |x|$ and $' = d/ds$. For small $\varepsilon > 0$ to be chosen later, we multiply this equation by $|u'|^\varepsilon$ and observe that the resulting equation can be written as:

$$-\frac{1}{1+\varepsilon}s^{-\tilde{N}+1}\big(s^{\tilde{N}-1}|u'|^\varepsilon u'\big)' + |u'|^{q+\varepsilon} \leq |f||u'|^\varepsilon, \tag{3.15}$$

where $\tilde{N} = 1 + (N-1)(1+\varepsilon) > N$. We now proceed as in Remark 2: we select a radially symmetric cut-off function $\xi \in C^\infty(B_{2R})$ such that $0 \leq \xi \leq 1$ and $\xi = 1$ in B_R. We multiply (3.15) by ξ^α, for $\alpha > 0$ to be chosen and find that

$$\begin{aligned}\frac{\alpha}{1+\varepsilon}\int_0^{2R} s^{\tilde{N}-1}|u'|^\varepsilon u'\xi^{\alpha-1}\xi' &+ \int_0^{2R} s^{\tilde{N}-1}|u'|^{q+\varepsilon}\xi^\alpha \\ &\leq \int_0^{2R} s^{\tilde{N}-1}|f||u'|^\varepsilon\xi^\alpha \leq \int_0^{2R} s^{\tilde{N}-1}|f||u'|^\varepsilon \\ &\leq \left(\int_0^{2R} s^{\tilde{N}-1}|f|^{\theta'}\right)^{\frac{1}{\theta'}}\left(\int_0^{2R} s^{\tilde{N}-1}|u'|^{\varepsilon\theta}\right)^{\frac{1}{\theta}},\end{aligned}$$

where $\theta > 1$ and we have used Hölder's inequality. We take $\theta = q/\varepsilon$ and use the estimates obtained in Remark 2 to arrive at

$$\int_0^{2R} s^{\tilde{N}-1}|u'|^{q+\varepsilon}\xi^\alpha \leq C|f|_{L^{\frac{q}{q-\varepsilon}}(B_{2R})} + C\int_0^{2R} s^{\tilde{N}-1}|u'|^{\varepsilon+1}\xi^{\alpha-1} \tag{3.16}$$

and we are also using that

$$\int_0^{2R} s^{\tilde{N}-1}|f|^{\frac{q}{q-\varepsilon}} \leq C\int_0^{2R} s^{N-1}|f|^{\frac{q}{q-\varepsilon}}$$

since $\tilde{N} > N$. We choose next $\varepsilon \leq q(r-1)/r$ and use again Hölder's inequality in (3.16):

$$\int_0^{2R} s^{\tilde{N}-1}|u'|^{q+\varepsilon}\xi^\alpha \leq C + C\left(\int_0^{2R} s^{\tilde{N}-1}|u'|^{q+\varepsilon}\xi^{(\alpha-1)\frac{q+\varepsilon}{1+\varepsilon}}\right)^{\frac{1+\varepsilon}{q+\varepsilon}},$$

where C depends additionally on $|f|_{L^r(B_{2R})}$. Choosing $\alpha = \frac{q+\varepsilon}{q-1}$ and recalling that $\xi \equiv 1$ in B_R, we obtain:

$$\int_0^R s^{\tilde{N}-1}|u'|^{q+\varepsilon} \leq C. \tag{3.17}$$

Our intention is to obtain estimate (3.14) from this inequality. We first observe that, thanks to Hölder's inequality, if $\delta > 0$, we have

$$\begin{aligned}\int_0^R s^{N-1}|u'|^{q+\delta} &= \int_0^R s^{-\nu} s^{N-1+\nu}|u'|^{q+\delta}\\ &\le \left(\int_0^{2R} s^{-\nu\gamma'}\right)^{\frac{1}{\gamma'}} \left(\int_0^R s^{(N-1+\nu)\gamma}|u'|^{(q+\delta)\gamma}\right)^{\frac{1}{\gamma}},\end{aligned}$$

where $\nu > 0$ and $\gamma > 1$ are to be chosen next. Observe that if ν and γ are taken verifying

$$\begin{aligned}&\nu\gamma' < 1\\ &(N-1+\nu)\gamma \ge (N-1)(1+\varepsilon)\\ &(q+\delta)\gamma \le q+\varepsilon\end{aligned} \tag{3.18}$$

then (3.14) will follow thanks to (3.17). Choose γ such that the second equation in (3.18) holds with equality, that is:

$$\gamma = \frac{(N-1)(1+\varepsilon)}{(N-1+\nu)}.$$

In order to have $\gamma > 1$, we need to restrict ν to have $\nu < \varepsilon(N-1)$. The first equation in (3.18) is then equivalent to $\nu < \frac{\varepsilon(N-1)}{1+(N-1)(1+\varepsilon)}$ (which implies in particular $\nu < \varepsilon(N-1)$). Thus we choose $\nu = \tau\varepsilon$, with

$$\tau < \frac{N-1}{1+(N-1)(1+\varepsilon)}. \tag{3.19}$$

Finally, the third equation in (3.18) can be achieved with a small δ if $q < \frac{\varepsilon}{\gamma-1}$, that is, if

$$q < \frac{N-1+\tau\varepsilon}{N-1-\tau}. \tag{3.20}$$

Since $1 < q < N/(N-1)$, we can choose a small ε and τ in such a way that both (3.19) and (3.20) hold. Hence we have (3.18) and this finally shows (3.14) if δ is small enough. □

4. Proof of the main theorems

This section will be devoted to the proof of Theorems 1, 2, 3 and 4. We will consider first the proofs of existence of solutions, which are really only different in the use of the corresponding theorems in Section 3.

Proof of Theorem 1. We follow the same procedure as in [3] or [4]. Choose a sequence $\{f_n\}_{n=1}^{\infty} \subset C^\infty(\mathbb{R}^N)$ such that $f_n \to f$ in $L^1_{\rm loc}(\mathbb{R}^N)$ and for every nonnegative integer n consider the problem

$$\begin{cases} -\Delta u + |u|^{p-1}u + |\nabla u|^q = f_n & \text{in } B_n\\ u = 0 & \text{on } \partial B_n.\end{cases} \tag{4.1}$$

According to Theorem 5, there exists a unique solution $u_n \in C^2(B_n) \cap C^1(\bar{B}_n)$ to (4.1).

Choose s such that $q < s < \frac{2p}{p+1}$. By means of Theorem 6, for every $R \in (0, n/2)$ there exists a constant $C > 0$ depending on R such that

$$\int_{B_R} |u_n|^p + |\nabla u_n|^s \le C\left(\int_{B_{2R}} |f_n| + 1\right) \le C. \tag{4.2}$$

Since $\dfrac{2p}{p+1} < p$, we obtain bounds in $W^{1,s}(B_R)$ so that, after passing to a subsequence and by means of a diagonal procedure, we obtain

$$u_n \rightharpoonup u \quad \text{weakly in } W^{1,s}_{\text{loc}}(\mathbb{R}^N).$$

In particular, we may assume

$$\begin{aligned} u_n \to u &\quad \text{in } L^s_{\text{loc}}(\mathbb{R}^N) \\ u_n \to u &\quad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

Our intention is to prove that u is a solution to (1.2). Let us first check that $u \in W^{1,1}_{\text{loc}}(\mathbb{R}^N)$ and $\nabla u_n \to \nabla u$ in $L^1_{\text{loc}}(\mathbb{R}^N)^N$.

Let $h_n = f_n - |u_n|^{p-1}u_n - |\nabla u_n|^q$. The sequence $\{h_n\}$ is bounded in $L^1_{\text{loc}}(\mathbb{R}^N)$ by (4.2) and $-\Delta u_n = h_n$. Take $\varepsilon > 0$ and $\xi \in C^\infty(B_{2R})$ with $0 \le \xi \le 1$ and $\xi \equiv 1$ in B_R. Define

$$\psi(s) = \begin{cases} \inf\{s, \varepsilon\}, & s \ge 0 \\ -\psi(-s), & s \le 0. \end{cases}$$

Taking $\xi\psi(u_n - u_m)$ as a test function in the weak formulation of $-\Delta(u_n - u_m) = h_n - h_m$, we obtain

$$\int_{B_R \cap A_{n,m,\varepsilon}} |\nabla(u_n - u_m)|^2 \le \varepsilon \int_{B_{2R}} (|h_n| + |h_m|) + C\varepsilon \int_{B_{2R}} (|\nabla u_n| + |\nabla u_m|) \le C\varepsilon,$$

where $A_{n,m,\varepsilon} = \{x \in \mathbb{R}^N : \ |u_n(x) - u_m(x)| \le \varepsilon\}$. In addition:

$$\begin{aligned} \int_{B_R} |\nabla(u_n - u_m)| &= \int_{B_R \cap A_{n,m,\varepsilon}} |\nabla(u_n - u_m)| + \int_{B_R \cap A^c_{n,m,\varepsilon}} |\nabla(u_n - u_m)| \\ &\le |B_R|^{\frac{1}{2}} \Big(\int_{B_R \cap A_{n,m,\varepsilon}} |\nabla(u_n - u_m)|^2 \Big)^{\frac{1}{2}} + |B_R \cap A^c_{n,m,\varepsilon}|^{\frac{1}{q'}} \Big(\int |\nabla(u_n - u_m)|^q \Big)^{\frac{1}{q}} \\ &\le C\varepsilon^{\frac{1}{2}} + C\,|B_R \cap A^c_{n,m,\varepsilon}|^{\frac{1}{q'}}, \end{aligned}$$

where $A^c_{n,m,\varepsilon}$ denotes the complementary of $A_{n,m,\varepsilon}$. Since $u_n \to u$ in measure, $|B_R \cap A^c_{n,m,\varepsilon}| \to 0$, so that ∇u_n is a Cauchy sequence in $L^1_{\text{loc}}(\mathbb{R}^N)^N$, and we obtain $\nabla u_n \to w$ in $L^1_{\text{loc}}(\mathbb{R}^N)^N$. Of course, this gives $u \in W^{1,1}_{\text{loc}}(\mathbb{R}^N)$ with $w = \nabla u$.

Next recall that $|\nabla u_n|$ is bounded in $L^s_{\text{loc}}(\mathbb{R}^N)$ for every $s \in \big(0, \frac{2p}{p+1}\big)$, so that, owing to a consequence of Vitali's theorem, we deduce

$$\nabla u_n \to \nabla u \quad \text{in } L^s_{\text{loc}}(\mathbb{R}^N)^N$$

for every $s \in \big(0, \frac{2p}{p+1}\big)$, in particular for $s = q$.

Finally, set $g_n = f_n - |\nabla u_n|^q$, $g = f - |\nabla u|^q$, so that $g_n \to g$ in $L^1_{\text{loc}}(\mathbb{R}^N)$. Since $-\Delta(u_n - u_m) + |u_n|^{p-1}u_n - |u_m|^{p-1}u_m = g_n - g_m$ for arbitrary $n, m \in \mathbb{N}$, we may employ Kato's inequality (cf. the Appendix in [6]) to arrive at

$$-\Delta|u_n - u_m| + \big||u_n|^{p-1}u_n - |u_m|^{p-1}u_m\big| \le |g_n - g_m| \quad \text{in } \mathbb{R}^N.$$

On multiplying this inequality by ξ we have

$$\int_{B_R} \big||u_n|^{p-1}u_n - |u_m|^{p-1}u_m\big| \le \int_{B_{2R}} |g_n - g_m| + C\int_{B_{2R}} |u_n - u_m| \to 0.$$

In particular

$$|u_n|^{p-1}u_n \to |u|^{p-1}u \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^N),$$

so that we may pass to the limit in the equation verified by u_n to obtain that u is a solution to (1.2). It is a consequence of this proof that $u \in W^{1,s}_{\rm loc}(\mathbb{R}^N)$ for every $s \in \left(1, \frac{2p}{p+1}\right)$ and $u \in L^p_{\rm loc}(\mathbb{R}^N)$. Finally, since $\Delta u \in L^1_{\rm loc}(\mathbb{R}^N)$, we also have $u \in W^{1,s}_{\rm loc}(\mathbb{R}^N)$ for every $s \in (0, \frac{N}{N-1})$. □

Remark 3. When $f \in L^r_{\rm loc}(\mathbb{R}^N)$ for some $r > N$ and is nonnegative, a solution u can be constructed verifying also $u \in C^1(\mathbb{R}^N) \cap W^{2,r}_{\rm loc}(\mathbb{R}^N)$, provided also that $0 < q \le 1$. Indeed, if the sequence $\{f_n\}$ is chosen to converge to f in $L^r_{\rm loc}(\mathbb{R}^N)$, and u_n is the unique solution to (4.1) (which is nonnegative), we have thanks to (3.6) in Remark 1:

$$\int_{B_R} u_n^{pr} \le C \left(\int_{B_{2R}} |f_n|^r + 1 \right) \le C.$$

Hence, passing to the limit we find that $u^p \in L^r_{\rm loc}(\mathbb{R}^N)$. Thus $-\Delta u + |\nabla u|^q = h$ in $\mathbb{R}^N$ for some $h \in L^r_{\rm loc}(\mathbb{R}^N)$. We claim that this yields $u \in C^1(\mathbb{R}^N) \cap W^{2,r}_{\rm loc}(\mathbb{R}^N)$.

To see the claim just notice that $u \in W^{1,s}_{\rm loc}(\mathbb{R}^N)$ for every s such that $1 \le s < \frac{N}{N-1}$, since $\Delta u \in L^1_{\rm loc}(\mathbb{R}^N)$. Then $|\nabla u|^q \in L^{\frac{s}{q}}_{\rm loc}(\mathbb{R}^N)$ so that $\Delta u \in L^{\theta_1}_{\rm loc}(\mathbb{R}^N)$, where $\theta_1 = \min\{r, s/q\}$. We may assume $\theta_1 = s/q$, for otherwise $u \in W^{2,r}_{\rm loc}(\mathbb{R}^N)$ and we are done since $W^{2,r}_{\rm loc}(\mathbb{R}^N) \subset C^1(\mathbb{R}^N)$ when $r > N$. If $\theta_1 < N$, the Sobolev embedding gives $|\nabla u|^q \in L^{\frac{N\theta_1}{q(N-\theta_1)}}_{\rm loc}(\mathbb{R}^N)$, so that $\Delta u \in L^{\theta_2}_{\rm loc}(\mathbb{R}^N)$, where $\theta_2 = \min\{r, \frac{N\theta_1}{q(N-\theta_1)}\}$. It is easily checked that $\theta_2 > \theta_1$. Continuing this way, we obtain an increasing sequence θ_k defined by

$$\theta_k = \min \left\{ r, \frac{N\theta_{k-1}}{q(N-\theta_{k-1})} \right\},$$

provided that $\theta_{k-1} < N$, and with the property that $u \in W^{2,\theta_k}_{\rm loc}(\mathbb{R}^N)$. It can be proved that there must exist k such that $\theta_k > N$, so that $u \in C^1(\mathbb{R}^N)$ by the Sobolev embedding and then also $u \in W^{2,r}_{\rm loc}(\mathbb{R}^N)$, by classical regularity.

Proof of Theorem 2. Take $f_n \in C^\infty(\mathbb{R}^N)$ such that $f_n \ge 0$ for every n and $f_n \to f$ in $L^r_{\rm loc}(\mathbb{R}^N)$. Consider again problem (4.1), which admits a unique solution $u_n \in C^2(B_n) \cap C^1(\bar{B}_n)$ by Theorem 5. Since $q > 1$ we also have $u_n \in C^3(B_n)$ by standard regularity.

The solution u_n is strictly positive, so that we may use Theorem 7 to obtain that

$$\sup_{B_R} \, (u_n + |\nabla u_n|) \le C, \tag{4.3}$$

where C depends on R and on $|f|_{L^r(B_{4R})}$. Arguing exactly as in the proof of Theorem 1 we obtain that –passing to a subsequence– $u_n \to u$ in $W^{1,s}_{\rm loc}(\mathbb{R}^N)$, for every $s > 1$, where u is a solution to (1.2) which is in addition nonnegative. Passing to the limit in (4.3) we also have $u \in W^{1,\infty}_{\rm loc}(\mathbb{R}^N)$, and it then follows that $\Delta u \in L^r_{\rm loc}(\mathbb{R}^N)$, so that $u \in W^{2,r}_{\rm loc}(\mathbb{R}^N)$. The Sobolev embedding implies then $u \in C^1(\mathbb{R}^N)$. Finally, the strong maximum principle gives $u > 0$ in $\mathbb{R}^N$. □

Proof of Theorem 3. It is only a minor variation of the existence proof in Theorem 1. We only have to choose radially symmetric functions f_n in (4.1), and notice that the unique solution u_n has to be radially symmetric. Hence Theorem 8 can be used and in particular the application of Vitali's theorem as in Theorem 1 implies that $|\nabla u_n|^q \to |\nabla u|^q$ in $L^1_{\rm loc}(\mathbb{R}^N)$, so that u is a nonnegative solution to (1.2). $\square$

To conclude our proofs, let us consider next uniqueness. An important part in the proof of uniqueness for problem (1.2) is played by the minimal (classical) solution U_R to the boundary blow-up problem

$$\left\{ \begin{array}{ll} -\Delta U + c\,U^p - d|\nabla U|^q = 0 & \text{in } B_R \\ U = \infty & \text{on } \partial B_R, \end{array} \right. \tag{4.4}$$

where $c, d > 0$, which was shown to exist in Corollary 13 in [1] (actually it is unique when $0 < q \le 1$). Let us quote an important property of this solution:

Lemma 9. *The solution U_R verifies*

$$U_R \to 0 \quad \textit{uniformly on compact sets of } \mathbb{R}^N$$

when $R \to \infty$.

Proof. The solution U_R is clearly radially symmetric. Hence it verifies

$$\left\{ \begin{array}{ll} -(r^{N-1}U_R')' + r^{N-1}(cU_R^p - d|U_R'|^q) = 0 & \text{in } 0 < r < R \\ U_R(0) = U_{0,R},\ U_R'(0) = 0, & \end{array} \right.$$

for some $U_{0,R} > 0$. It follows by comparison that U_R is decreasing in R, and then it can also be proved that U_R' is decreasing in R (observe that $U_R' \ge 0$). This gives bounds for both U_R and U_R', and then it is standard to obtain that $U_R \to U$ when $R \to \infty$, uniformly on compacts, where U is a (radial) solution to

$$-\Delta U + cU^p - d|\nabla U|^q = 0 \quad \text{in } \mathbb{R}^N.$$

Take an arbitrary $x_0 \in \mathbb{R}^N$ and for arbitrary $R > 0$ consider the function $V_R(x) = U_R(x - x_0)$. It is clear that V_R solves the problem

$$\left\{ \begin{array}{ll} -\Delta V + cV^p - d|\nabla V|^q = 0 & \text{in } B_R(x_0) \\ V = \infty & \text{on } \partial B_R(x_0), \end{array} \right.$$

and thus by comparison we have $U \le V_R$ in $B_R(x_0)$, since $U < \infty$ on $\partial B_R(x_0)$. Letting $R \to \infty$ we arrive at $U(x) \le U(x - x_0)$ in $\mathbb{R}^N$. The arbitrariness of x_0 implies that U is constant, hence $U \equiv 0$. $\square$

Now we can finally conclude the proofs of our results. We remark that the cases $0 < q \le 1$ and $1 < q < p$ are quite different.

Proof of Theorem 4. Assume first $0 < q \le 1$. Observe that by Remark 3 we have a nonnegative solution $u \in C^1(\mathbb{R}^N)$ to (1.2). Let $v \in W^{1,\infty}_{\rm loc}(\mathbb{R}^N)$ be another solution to (1.2). By the same discussion as in the proof of Theorem 2 we obtain $v \in C^1(\mathbb{R}^N)$.

We have $-\Delta(u - v) + |u|^{p-1}u - |v|^{p-1}v + |\nabla u|^q - |\nabla v|^q = 0$ in $\mathbb{R}^N$. Observe on one hand that

$$\big||\nabla u|^q - |\nabla v|^q\big| \le \big|\nabla(u - v)\big|^q \tag{4.5}$$

and on the other hand that

$$\big||u|^{p-1}u - |v|^{p-1}v\big| \ge c|u - v|^{p-1}(u - v) \tag{4.6}$$

for some positive constant c. Thus setting $z = u - v$ we have $-\Delta z + c|z|^{p-1}z - |\nabla z|^q \le 0$ in $\mathbb{R}^N$.

If U_R denotes the (unique) solution to (4.4) with $d = 1$, we have $z < U_R$ near ∂B_R. By using the comparison principle (cf. for instance Lemma 2.1 in [8]) we arrive at

$$z \le U_R \quad \text{in } B_R.$$

We now let $R \to \infty$ and use Lemma 9 to obtain that $z \le 0$ in $\mathbb{R}^N$, that is, $u \le v$ in $\mathbb{R}^N$. The symmetric argument then gives $u = v$ and this shows uniqueness when $0 < q \le 1$.

Now consider the case $1 < q < p$. We notice that (4.5) is no longer valid. However, for fixed small $\delta > 0$ there exists a positive constant $C(\delta)$ such that

$$\big|(1+\delta)a - b\big|^q \ge (1+\delta)a^q - C(\delta)b^q \qquad \text{when } a, b > 0.$$

Let U_R be the minimal solution to (4.4) with $d = C(\delta)$ and c as in (4.6).

Let $u \in C^1(\mathbb{R}^N)$ be the solution to (1.2) constructed in Theorem 2. We claim that $\bar{u} = (1+\delta)u + U_R$ is a supersolution to (1.2). Indeed:

$$\begin{aligned} -\Delta\bar{u} + \bar{u}^p + |\nabla\bar{u}|^q &= -(1+\delta)\Delta u - \Delta U_R + \big((1+\delta)u + U_R\big)^p + \big|(1+\delta)\nabla u + \nabla U_R\big|^q \\ &= -(1+\delta)u^p - (1+\delta)|\nabla u|^q - cU_R^p + c(\delta)|\nabla U_R|^q + f \\ &\quad + \big((1+\delta)u + U_R\big)^p + \big|(1+\delta)\nabla u + \nabla U_R\big|^q. \end{aligned}$$

But $\big|(1+\delta)\nabla u + \nabla U_R\big|^q \ge \big|(1+\delta)|\nabla u| - |\nabla U_R|\big|^q \ge (1+\delta)|\nabla u|^q - C(\delta)|\nabla U_R|^q$ and $\big((1+\delta)u + U_R\big)^p \ge (1+\delta)^p u^p + cU_R^p \ge (1+\delta)u^p + cU_R^p$, so that $-\Delta\bar{u} + \bar{u}^p + |\nabla\bar{u}|^q \ge f$.

Next, observe that for every solution $v \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N)$ we have $\Delta v \in L^r_{\text{loc}}(\mathbb{R}^N)$, hence $v \in W^{2,r}_{\text{loc}}(\mathbb{R}^N)$ and by the Sobolev embedding $v \in C^1(\mathbb{R}^N)$. In particular, we have $v < \bar{u}$ near ∂B_R and it follows by comparison as before that

$$v \le (1+\delta)u + U_R \quad \text{in } B_R$$

for every $R > 0$. Letting first $R \to \infty$, using Lemma 9 and then allowing $\delta \to 0$, we arrive at $v \le u$.

It can be proved in a similar way that $(1-\delta)u - U_R$ is a subsolution to (1.2), and a comparison as before yields $(1-\delta)u - U_R \le v$ in B_R. Letting $R \to \infty$ and then $\delta \to 0$ we obtain $u = v$, which proves uniqueness. □

Remark 4. When $q \ge p$, uniqueness of $W^{1,\infty}_{\text{loc}}(\mathbb{R}^N)$ solutions does not hold. This can be seen by taking $f = 0$, where aside from the trivial solution, there are infinitely many negative radial (smooth) solutions. Indeed, if we set $v = -u$, we look for radial positive solutions to $-\Delta v + v^p - |\nabla v|^q = 0$ in $\mathbb{R}^N$, which verify the Cauchy problem

$$\begin{cases} (r^{N-1}v')' + r^{N-1}(v^p - |v'|^q) = 0 \\ v(0) = v_0, \quad v'(0) = 0, \end{cases}$$

for some $v_0 > 0$. The solutions of this problem are defined in an interval $[0, T)$, and from Proposition 3 in [1] we know that $v' > 0$, $v'' \ge 0$, so that necessarily $\lim_{r\to T-} v(r) = +\infty$. However, if $T < \infty$ then v would be a solution to

$$\begin{cases} \Delta v = v^p - |\nabla v|^q & \text{in } B_T \\ v = \infty & \text{on } \partial B_T \end{cases}$$

which contradicts Corollary 13 in [1], since $p \le q$. Hence $T = \infty$ and this shows that $u = -v$ is a solution to $-\Delta u + |u|^{p-1}u + |\nabla u|^q = 0$ in $\mathbb{R}^N$.

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